

On Dynamic Deviation Measures and Continuous-Time Portfolio Optimisation

Martijn Pistorius*

Mitja Stadje†

Abstract. In this paper we propose the notion of *dynamic deviation measure*, as a dynamic time-consistent extension of the (static) notion of deviation measure. To achieve time-consistency we require that a dynamic deviation measure satisfies a generalised conditional variance formula. We show that, under a domination condition, dynamic deviation measures are characterised as the solutions to a certain class of backward SDEs. We establish for any dynamic deviation measure an integral representation, and derive a dual characterisation result in terms of *additively m -stable* dual sets. Using this notion of dynamic deviation measure we formulate a dynamic mean-deviation portfolio optimisation problem in a jump-diffusion setting and identify a subgame-perfect Nash equilibrium strategy that is linear as function of wealth by deriving and solving an associated extended HJB equation.

1 Introduction

One traditional way of thinking about risk is in terms of the extent that random realisations deviate from the mean. In portfolio theory as initiated in Markowitz (1952), for instance, risk is quantified as the variance or standard deviation of the return. In the setting of the Black-Scholes (1973) model, it is the volatility parameter, which is equal to the standard deviation of the log-stock price at unit time, that is often taken as description of the risk. Alternative approaches to quantification of risk that have emerged more recently also take into account other aspects of the return distribution such as heavy tails and asymmetry. In this context an axiomatic framework for (general) deviation measures was introduced and developed in Rockafellar *et al.* (2006a), which form a certain class of non-negative positively homogeneous (static) operators acting on square-integrable random variables. General deviation measures allow to distinguish between upper and lower deviations from the mean, generalising standard deviation. Various aspects of portfolio optimisation and financial decision making under general deviation measures have been explored in the literature, in particular regarding CAPM, asset betas, one- and two-fund theorems and equilibrium theory; see also among many others Cheng *et al.* (2004), Rockafellar *et al.* (2006b, 2006c, 2007), Märket and Schultz (2005), Stoyanov *et al.* (2008), Grechuk *et al.* (2009), or Grechuk and Zabarankin (2013, 2014). In this paper we present an axiomatic approach to deviation measures in *dynamic* continuous-time settings. We show that such dynamic deviation measures admit in general a dual robust representation and are linked to a certain family of backward stochastic differential equations (BSDEs), if a certain domination condition is satisfied. Subsequently, we use this notion of dynamic deviation measure to phrase a mean-deviation portfolio optimisation problem in a jump-diffusion setting and identify for this problem a subgame-perfect Nash equilibrium portfolio allocation strategy by means of an associated novel type of extended Hamilton-Jacobi-Bellman equation, which complements the ones studied in Björk and Murgoci (2010).

*Department of Mathematics, Imperial College London, m.pistorius@imperial.ac.uk

†Faculty of Mathematics and Economics, University of Ulm, mitja.stadje@uni-ulm.de

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(Conditional) deviation measures. Dynamic deviation measures are given in terms of conditional deviation measures, which are in turn a conditional version of the notion of (static) deviation measure defined in Rockafellar *et al.* (2006a) that we describe next. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, where $T > 0$ denotes the horizon, consider the (risky) positions described by elements in $L^p(\mathcal{F}_t)$, $t \in [0, T]$, $p \geq 0$, the space of \mathcal{F}_t -measurable random variables X such that $\mathbb{E}[|X|^p] < \infty$; by $L_+^p(\mathcal{F}_t)$, $L^\infty(\mathcal{F}_t)$ and $L_+^\infty(\mathcal{F}_t)$ are denoted the subsets of non-negative, bounded and non-negative bounded elements in $L^p(\mathcal{F}_t)$. The definition is given as follows:

Definition 1.1 *For any given $t \in [0, T]$, $D_t : L^2(\mathcal{F}_T) \rightarrow L_+^2(\mathcal{F}_t)$ is called an \mathcal{F}_t -conditional deviation measure if it is normalised ($D_t(0) = 0$) and the following properties are satisfied:*

- (D1) Translation Invariance: $D_t(X + m) = D_t(X)$ for any $m \in L^\infty(\mathcal{F}_t)$;
- (D2) Positive Homogeneity: $D_t(\lambda X) = \lambda D_t(X)$ for any $X \in L^2(\mathcal{F}_T)$ and $\lambda \in L_+^\infty(\mathcal{F}_t)$;
- (D3) Subadditivity: $D_t(X + Y) \leq D_t(X) + D_t(Y)$ for any $X, Y \in L^2(\mathcal{F}_T)$;
- (D4) Positivity: $D_t(X) \geq 0$ for any $X \in L^2(\mathcal{F}_T)$, and $D_t(X) = 0$ if and only if X is \mathcal{F}_t -measurable.

If \mathcal{F}_0 is trivial, D_0 is a deviation measure in the sense of Definition 1 in Rockafellar *et al.* (2006a). The value $D_t(X) = 0$, we recall, corresponds to the riskless state of no uncertainty, and axiom (D1) can be interpreted as the requirement that adding to a position X a constant (interpreted as cash) should not increase the risk. Furthermore, it follows similarly as in Rockafellar *et al.* (2006a) that, if D satisfies (D2)–(D3), (D1) holds if and only if $D_t(m) = 0$ for any $m \in L^2(\mathcal{F}_t)$. In other words, constants do not carry any risk. Moreover, it is well known that if (D2) holds, (D3) is equivalent to *conditional convexity*, that is, for any $X, Y \in L^2(\mathcal{F}_T)$ and any $\lambda \in L^\infty(\mathcal{F}_t)$ that is such that $0 \leq \lambda \leq 1$

$$D_t(\lambda X + (1 - \lambda)Y) \leq \lambda D_t(X) + (1 - \lambda)D_t(Y).$$

The property of convexity is often given the interpretation that diversification of a position should not increase its riskiness. We also note that (D2) implies that, for any $X_1, X_2 \in L^2(\mathcal{F}_T)$, $D_t(I_A X_i) = I_A D_t(X_i)$, $i = 1, 2$, where I_A denotes the indicator of the set A , so that*

$$D_t(I_A X_1 + I_{A^c} X_2) = I_A D_t(X_1) + I_{A^c} D_t(X_2), \quad A \in \mathcal{F}_t. \quad (1.1)$$

In the analysis typically also a lower semi-continuity condition is imposed, the conditional version of which is given as follows:

- (D5) Lower Semi-Continuity: *If X^n converges to X in $L^2(\mathcal{F}_T)$ then $D_t(X) \leq \liminf_n D_t(X^n)$.*

Dynamic deviation measures. We impose additional structure on a given family of \mathcal{F}_t -conditional deviation measures in order to ensure it satisfies a form of time-consistency. One recursive structure that has been successfully deployed in among others the case of mean-variance portfolio optimisation is the one embedded in the *conditional variance formula*; see for instance Basak and Chabakauri (2010), Wang and Forsyth (2011), Li *et al.* (2012) or Czichowsky (2013). Inspired by this recursive structure we require that a collection $(D_t)_{t \in [0, T]}$ of conditional deviation measures satisfy the following generalisation of the conditional variance formula:

- (D6) Time-Consistency: *For all $s, t \in [0, T]$ with $s \leq t$ and $X \in L^2(\mathcal{F}_T)$*

$$D_s(X) = D_s(\mathbb{E}[X|\mathcal{F}_t]) + \mathbb{E}[D_t(X)|\mathcal{F}_s]. \quad (1.2)$$

*To see that (1.1) holds note that by (D2) $I_A D_t(1_A X_1 + 1_{A^c} X_2) = D_t(I_A(1_A X_1 + 1_{A^c} X_2)) = D_t(I_A X_1) = I_A D_t(X_1)$; similarly, we have $I_{A^c} D_t(1_A X_1 + 1_{A^c} X_2) = I_{A^c} D_t(X_2)$.

Remark 1.2 (i) As $D(X) \geq 0$, (D6) implies that $(D_s(X))_{s \in [0, T]}$ is a supermartingale, which implies in particular that D has a càdlàg modification.

(ii) It follows by standard arguments that (D6) for $s = 0$ already uniquely determines a dynamic deviation measure D . For suppose that D_0 and $X \in L^2(\mathcal{F}_T)$ are given and besides $(D_t(X))_{t \in [0, T]}$ there exists a collection of square-integrable \mathcal{F}_s -measurable random variables $(D'_t(X))_{t \in [0, T]}$ satisfying (D6) for $s = 0$, then $D_t(X) = D'_t(X)$ for all $t \in [0, T]$. Indeed, if the \mathcal{F}_t -measurable set $A' := \{D'_t(X) > D_t(X)\}$ were to have non-zero measure, then by (1.1) and (D6) we find

$$\mathbb{E}[I_{A'} D_t(X)] = \mathbb{E}[D_t(I_{A'} X)] = D_0(I_{A'} X) - D_0(\mathbb{E}[I_{A'} X | \mathcal{F}_t]) = \mathbb{E}[D'_t(I_{A'} X)] = \mathbb{E}[I_{A'} D'_t(X)],$$

which is a contradiction to the definition of the set A' . Similarly, it may be seen that the set $\{D'_t(X) < D_t(X)\}$ has measure zero.

(iii) Since D_0 is convex, lower semi-continuous and finite, D_0 is continuous in $L^2(\mathcal{F}_T)$ (see Proposition 2 in Rockafellar *et al.* (2006)).

We arrive thus at the following definition of dynamic deviation measure:

Definition 1.3 A family $(D_t)_{t \in [0, T]}$ is called a dynamic deviation measure if D_t , $t \in [0, T]$, are \mathcal{F}_t -conditional deviation measures satisfying (D5) and (D6).

One way to construct examples of dynamic deviation measures is in terms of the solutions of a certain type of BSDEs. Such solutions, when seen as function of the corresponding random variable, we will call g -deviation measures (where g is the driver function of the BSDE in question). We show in Theorem 3.2 that, under a domination condition, any dynamic deviation measure is equal to a g -deviation measure for some driver function g . This result may be considered to be an analogue of the link between the dynamic coherent and convex risk measures and g -expectations; see Coquet *et al.* (2002) and Royer (2006) (for contributions on convex risk measures and g -expectations and their generalizations see for instance Barrieu and El Karoui (2005, 2009), Rosazza Gianin (2006), Klöppel and Schweizer (2007), Jiang (2008), El Karoui and Ravenelli (2009), Bion-Nadal and Magali (2012) or Pelsser and Stadje (2014)). By drawing on dual robust representation results we also establish characterisations of general dynamic deviation measures that are valid without the domination condition (see Theorems 4.1, 4.3 and 4.4).

Remark 1.4 (Relation to dynamic coherent risk-measures) By generalising arguments given in Rockafellar *et al.* (2006) to the \mathcal{F}_t -conditional context, we note that any \mathcal{F}_t -conditional deviation measure is equal to the sum of a conditional expectation and a risk-measure ρ_t that satisfies a (\mathcal{F}_t -conditional) lower range dominance condition (that is, $\rho_t(X) \geq \mathbb{E}[X | \mathcal{F}_t]$ for all $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$ with equality on sets in \mathcal{F}_t on which X is constant). As the notions of time-consistency differ in cases of dynamic deviation and dynamic risk measures this relation does not carry over to the dynamic case. A collection $(\rho_t)_{t \in [0, T]}$, $\rho_t : L^2(\mathcal{F}_T) \rightarrow L^2_+(\mathcal{F}_t)$, forms a family of dynamic coherent risk measures, we recall, if, for every $t \in [0, T]$, ρ_t is positively homogeneous and subadditive (as in (D2) and (D3)), and is (dynamically) monotone and translation invariant in the following sense:

Translation Invariance: For all $X \in L^2(\mathcal{F}_T)$ and $m \in L^\infty(\mathcal{F}_t)$ we have $\rho_t(X + m) = \rho_t(X) - m$.

Monotonicity: If $X, Y \in L^2(\mathcal{F}_T)$ and $X \leq Y$ then $\rho_t(X) \geq \rho_t(Y)$.

For a discussion of these axioms see Artzner *et al.* (1999). Note that by (D1)–(D2) $D_t(m) = 0$ for any $m \in L^2(\mathcal{F}_t)$, so that dynamic deviation measures do not satisfy the axiom of monotonicity. While for dynamic deviation measures time-consistency is defined in terms of the generalised conditional variance formula (1.2), in the theory of dynamic coherent and convex risk measures a recursive tower-type

property is the relation strongly time-consistent dynamic risk-measures should satisfy. Specifically, a dynamic coherent or convex risk measures is called strongly time-consistent, we recall, if

$$\rho_s(\rho_t(X)) = \rho_s(X) \quad \text{for } s \leq t, \quad (1.3)$$

see for instance among many others Chen and Epstein (2002), Riedel (2004), Delbaen (2006), Artzner *et al.* (2007), Föllmer and Schied (2011), Cheridito and Kupper (2011). Note that a dynamic deviation measure D is not strongly time-consistent (in view of the fact that $D_t(D_T(X)) = D_t(0) = 0$ for $t < T$). Interestingly, as shown in Proposition 4.9, a collection of conditional deviation measures satisfies (D6) if and only if in their dual representations the dual sets are convex, closed, and *additively* m -stable, which is a result naturally complementing the well-known fact in the literature that the property of time-consistency for coherent risk measures (defined by (1.3)) may be characterised in terms of convex, closed, *multiplicatively* m -stable sets (see Delbaen (2006)).

Contents. The remainder of the paper is organised as follows. We present in Section 2 the definition of g -deviation measures, its properties and a number of examples. With these results in hand, we turn in Section 3 to the characterisation of dynamic deviation measures under a domination condition (Theorem 3.2). and proceed to establish in Section 4 an integral representation for general dynamic deviation measures, removing the aforementioned domination condition, (Theorem 4.1) and a dual robust representation result. (Theorems 4.3 and 4.4). In Section 5 we phrase a dynamic mean-deviation portfolio-optimisation problem and present an equilibrium solution. It is of interest to investigate other (financial) optimisation problems in terms of dynamic deviation measures, such as optimal hedging problems, capital allocation problems and optimal stopping problems; in the interest of brevity, we leave these as topics for future research.

2 g -deviation measures

In the sequel we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with (i) a standard d -dimensional Brownian motion $W = (W^1, \dots, W^d)^\top$ and (ii) a Poisson random measure $N(dt \times dx)$ on $[0, T] \times \mathbb{R}^k \setminus \{0\}$, independent of W , with intensity measure $\hat{N}(dt \times dx) = \nu(dx)dt$, where the Lévy measure $\nu(dx)$ satisfies the integrability condition

$$\int_{\mathbb{R}^k \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty,$$

and let $\tilde{N}(dt \times dx) := N(dt \times dx) - \hat{N}(dt \times dx)$ denote the compensated Poisson random measure. Further, let \mathcal{U} denote the Borel sigma-algebra induced by the $L^2(\nu(dx))$ -norm, $(\mathcal{F}_t)_{t \in [0, T]}$ the right-continuous completion of the filtration generated by W and N , and \mathcal{P} and \mathcal{O} the predictable and optional sigma-algebras on $[0, T] \times \Omega$ with respect to (\mathcal{F}_t) . We denote by $L_d^2(\mathcal{P}, d\mathbb{P} \times dt)$ the space of all predictable d -dimensional processes that are square-integrable with respect to the measure $d\mathbb{P} \times dt$ and we let $\mathcal{S}^2 = \{Y \in \mathcal{O} : \mathbb{E}[\sup_{0 \leq t \leq T} |Y_s|^2] < \infty\}$ denote the collection of square-integrable càdlàg optional processes. Further, let $\mathcal{B}(\mathbb{R}^k \setminus \{0\})$ be the Borel sigma-algebra on $\mathbb{R}^k \setminus \{0\}$. For any $X \in L^2(\mathcal{F}_T)$ we denote by (H^X, \tilde{H}^X) the unique pair of predictable processes with $H^X \in L_d^2(\mathcal{P}, d\mathbb{P} \times dt)$ and $\tilde{H}^X \in L^2(\mathcal{P} \times \mathcal{B}(\mathbb{R}^k \setminus \{0\}), d\mathbb{P} \times dt \times \nu(dx))$, subsequently referred to as the *representing pair* of X , satisfying[†]

$$X = \mathbb{E}[X] + \int_0^T H_s^X dW_s + \int_0^T \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s^X(x) \tilde{N}(dt \times dx), \quad (2.1)$$

where $\int_0^T H_s^X dW_s := \sum_{i=1}^d \int_0^T H_s^{X,i} dW_s^i$.

We consider the following class of driver functions:

[†]See *e.g.* Theorem III.4.34 in Jacod and Shiryaev (2013)

Definition 2.1 We call a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable function

$$g : \begin{matrix} [0, T] & \times & \Omega & \times & \mathbb{R}^d & \times & L^2(\nu(dx)) \\ (t, & & \omega, & & h, & & \tilde{h}) \end{matrix} \begin{matrix} \rightarrow & \mathbb{R}_+ \\ \mapsto & g(t, \omega, h, \tilde{h}) \end{matrix}$$

a driver function if for $d\mathbb{P} \times dt$ a.e. $(\omega, t) \in \Omega \times [0, T]$:

- (i) (Positivity) For any $(h, \tilde{h}) \in \mathbb{R}^d \times L^2(\nu(dx))$ $g(t, h, \tilde{h}) \geq 0$ with equality if and only if $(h, \tilde{h}) = 0$.
- (ii) (Lower semi-continuity) If $h^n \rightarrow h$, $\tilde{h}^n \rightarrow \tilde{h}$ $L^2(\nu(dx))$ -a.e. then $g(t, h, \tilde{h}) \leq \liminf_n g(t, h^n, \tilde{h}^n)$.

Definition 2.2 We call a driver function g convex if $g(t, h, \tilde{h})$ is convex in (h, \tilde{h}) , $d\mathbb{P} \times dt$ a.e.; positively homogeneous if $g(t, h, \tilde{h})$ is positively homogeneous in (h, \tilde{h}) , i.e., for $\lambda > 0$, $g(t, \lambda h, \lambda \tilde{h}) = \lambda g(t, h, \tilde{h})$, $d\mathbb{P} \times dt$ a.e. and of linear growth if for some $K > 0$ we have $d\mathbb{P} \times dt$ a.e.

$$|g(t, h, \tilde{h})|^2 \leq 1 + K^2|h|^2 + K^2 \int_{\mathbb{R}^k \setminus \{0\}} \tilde{h}(x)^2 \nu(dx). \quad (2.2)$$

To such a driver function g one may associate a corresponding dynamic deviation measure given in terms of the solution to a certain BSDE.

Definition 2.3 Let g be a convex and positively homogeneous driver function of linear growth. The g -deviation measure $D^g = (D_t^g)_{t \in [0, T]}$ is equal to the collection $D_t : L^2(\mathcal{F}_T) \rightarrow L^2_+(\mathcal{F}_t)$, $t \in [0, T]$, given by

$$D_t^g(X) = Y_t, \quad X \in L^2(\mathcal{F}_T),$$

where $(Y, Z, \tilde{Z}) \in \mathcal{S}^2 \times L^2_d(\mathcal{P}, d\mathbb{P} \times dt) \times L^2(\mathcal{P} \times \mathcal{B}(\mathbb{R}^k \setminus \{0\}), d\mathbb{P} \times dt \times \nu(dx))$ is the unique solution of the BSDE given in terms of the representing pair (H^X, \tilde{H}^X) of X by

$$dY_t = -g(t, H_t^X, \tilde{H}_t^X)dt + Z_t dW_t + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{Z}_t(x) \tilde{N}(dt \times dx), \quad t \in [0, T], \quad (2.3)$$

$$Y_T = 0, \quad (2.4)$$

Any g -deviation measure admits an integral representation in terms of g .

Proposition 2.4 Let g be a convex and positively homogeneous driver function of linear growth.

(i) For given $X \in L^2(\mathcal{F}_T)$, we have

$$D_t^g(X) = \mathbb{E} \left[\int_t^T g(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \quad (2.5)$$

(ii) D^g is a dynamic deviation measure. In particular, D^g satisfies (D6).

Proof. (i) Letting Y_t be equal to the right-hand side of (2.5) we note that $Y_T = 0$, while we have

$$Y_t = M_t - \int_0^t g(s, H_s^X, \tilde{H}_s^X) ds, \quad M_t = \mathbb{E} \left[\int_0^T g(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t \right].$$

Letting $(Z, \tilde{Z}) = (Z^{M_T}, \tilde{Z}^{M_T})$ the representing pair of M_T we have that Y_t satisfies (2.3).

(ii) To verify that (D6) holds we note that the representation (2.5) implies that, for any $s, t \in [0, T]$ with $s \leq t$,

$$D_s^g(\mathbb{E}[X | \mathcal{F}_t]) = \mathbb{E} \left[\int_s^t g(u, H_u^X, \tilde{H}_u^X) du \middle| \mathcal{F}_s \right],$$

which yields that $D_s^g(\mathbb{E}[X|\mathcal{F}_t]) + \mathbb{E}[D_t^g(X)|\mathcal{F}_s]$ is equal to

$$\mathbb{E}\left[\int_s^t g(u, H_u^X, \tilde{H}_u^X) du \middle| \mathcal{F}_s\right] + \mathbb{E}\left[\mathbb{E}\left[\int_t^T g(u, H_u^X, \tilde{H}_u^X) du \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^T g(u, H_u^X, \tilde{H}_u^X) du \middle| \mathcal{F}_s\right],$$

which is equal to $D_s^g(X)$. We show next that the axioms (D1)–(D5) are satisfied. We note from (2.5) that $D_t^g(X+m) = D_t^g(X)$ for any $X \in L^2(\mathcal{F}_T)$, $m \in L_+^\infty(\mathcal{F}_t)$ while $D_t^g(m) = 0$ as $g(t, 0, 0) = 0$, so that (D1) holds. Using (2.5) we see that D^g inherits the properties of convexity and positive homogeneity from g , so that (D2) and (D3) are satisfied. Positivity (D4) is straightforward to verify by using that g is nonnegative and strictly positive for $(h, \tilde{h}) \neq 0$. Finally, noting that (a) if $X^n \rightarrow X$ in $L^2(\mathcal{F}_T)$, $(H^{X^n}, \tilde{H}^{X^n})$ converges to (H^X, \tilde{H}^X) in $L_d^2(d\mathbb{P} \times dt) \times L^2(d\mathbb{P} \times dt \times \nu(dx))$ -norm and that (b) g is nonnegative and lower semi-continuous, we have by an application of Fatou's Lemma

$$\begin{aligned} \liminf_n D_t^g(X^n) &= \liminf_n \mathbb{E}\left[\int_t^T g(s, H_s^{X^n}, \tilde{H}_s^{X^n}) ds \middle| \mathcal{F}_t\right] \geq \mathbb{E}\left[\int_t^T \liminf_n g(s, H_s^{X^n}, \tilde{H}_s^{X^n}) ds \middle| \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[\int_t^T g(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t\right] = D_t^g(X), \end{aligned}$$

which shows that also the lower-semicontinuity condition in (D5) is satisfied. \square

The linear growth condition and convexity guarantee that a g -deviation measure is continuous in L^2 .

Lemma 2.5 *Let g be a convex driver function of linear growth. If X^n converge to X in $L^2(\mathcal{F}_T)$ then $\lim_n D_0^g(X^n) = D_0^g(X)$.*

Proof. If X^n converge to X in $L^2(\mathcal{F}_T)$ then, as noted before, H^{X^n} and \tilde{H}^{X^n} converge to H^X and \tilde{H}^X in $L_d^2(d\mathbb{P} \times dt)$ and $L^2(d\mathbb{P} \times dt \times \nu(dx))$ norms. Next note that $|g(s, H_s^{X^n}, \tilde{H}_s^{X^n})|$ is a uniformly integrable sequence by the growth-condition (2.2) and the convergence of the processes $|H^{X^n}|^2$ and $\int_{\mathbb{R}^k \setminus \{0\}} |\tilde{H}^{X^n}|^2(x) \nu(dx)$ in $L^1(d\mathbb{P} \times dt)$ -norm. As g is continuous (as it is convex and locally bounded, cf. Theorem 2.2.9 in Zalescu (2002)) it follows thus that $\lim_n D_0^g(X^n) = \lim_n \mathbb{E}\left[\int_0^T g(s, H_s^{X^n}, \tilde{H}_s^{X^n}) ds\right] = \mathbb{E}\left[\int_0^T g(s, H_s^X, \tilde{H}_s^X) ds\right] = D_0^g(X)$. \square

We list a number of properties of a g -deviation measure that are characterised in terms of those of the driver function g .

Proposition 2.6 *Let g and \tilde{g} be driver functions of linear growth.*

- (i) D^g is conditionally convex if and only if g is convex.
- (ii) D^g satisfies (D2) if and only if g is positively homogeneous.
- (iii) D^g is symmetric, that is, $D_t^g(X) = D_t^g(-X)$ for all t , if and only if g is symmetric in (h, \tilde{h}) .
- (iv) $D^g \geq D^{\tilde{g}}$ if and only if $g \geq \tilde{g}$ $d\mathbb{P} \times dt$ a.e.

To simplify notation we denote, for $s, t \in [0, T]$ with $s \leq t$ and $(H, \tilde{H}) \in L_d^2(\mathcal{P}, d\mathbb{P} \times dt \times L^2(\mathcal{P} \times \mathcal{B}(\mathbb{R}^k \setminus \{0\}), d\mathbb{P} \times dt \times \nu(dx)))$, $(H \cdot W)_{s,t} := \int_s^t H_u dW_u$ and $(\tilde{H} \cdot \tilde{N})_{s,t} := \int_{(s,t] \times \mathbb{R}^k \setminus \{0\}} \tilde{H}_u(x) \tilde{N}(du \times dx)$, and moreover $(H \cdot W)_t := (H \cdot W)_{0,t}$ and $(\tilde{H} \cdot \tilde{N})_t := (\tilde{H} \cdot \tilde{N})_{0,t}$.

Proof of Proposition 2.6. First, we prove (i) \Rightarrow by contradiction. Suppose that there exist predictable processes B^i and \tilde{B}^i for $i = 1, 2$, a nonzero predictable set C and a $\lambda \in (0, 1)$ such that for $(s, \omega) \in C$

$$g(s, \lambda B_s^1 + (1 - \lambda) B_s^2, \lambda \tilde{B}_s^1 + (1 - \lambda) \tilde{B}_s^2) > \lambda g(s, B_s^1, \tilde{B}_s^1) + (1 - \lambda) g(s, B_s^2, \tilde{B}_s^2).$$

Set $H_s^i(\omega) = B_s^i(\omega)$, $i = 1, 2$, if $(s, \omega) \in C$ and zero otherwise, define \tilde{H}^i , $i = 1, 2$, similarly and set $X = (H^1 \cdot W)_T + (\tilde{H}^1 \cdot \tilde{N})_T$, $Y = (H^2 \cdot W)_T + (\tilde{H}^2 \cdot \tilde{N})_T$ and $C_s = \{\omega \in \Omega : (s, \omega) \in C\}$. using that $g(s, 0, 0) = 0$ it follows that $D_0^g(\lambda X + (1 - \lambda)Y)$ is equal to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T g(s, \lambda I_{C_s} H_s^1 + (1 - \lambda) I_{C_s} H_s^2, \lambda I_{C_s} \tilde{H}_s^1 + (1 - \lambda) I_{C_s} \tilde{H}_s^2) ds \right] \\ &= \mathbb{E} \left[\int_0^T I_{C_s} g(s, \lambda H_s^1 + (1 - \lambda) H_s^2, \lambda \tilde{H}_s^1 + (1 - \lambda) \tilde{H}_s^2) ds \right] \\ &> \lambda \mathbb{E} \left[\int_0^T I_{C_s} g(s, H_s^1, \tilde{H}_s^1) ds \right] + (1 - \lambda) \mathbb{E} \left[\int_0^T I_{C_s} g(s, H_s^2, \tilde{H}_s^2) ds \right] \\ &= \lambda \mathbb{E} \left[\int_0^T g(s, I_{C_s} H_s^1, I_{C_s} \tilde{H}_s^1) ds \right] + (1 - \lambda) \mathbb{E} \left[\int_0^T g(s, I_{C_s} H_s^2, I_{C_s} \tilde{H}_s^2) ds \right]. \end{aligned} \quad (2.6)$$

The right-hand side of (2.6) is equal to $\lambda D_0^g(X) + (1 - \lambda) D_0^g(Y)$, in contradiction to the convexity of D_0^g . The directions ‘ \Rightarrow ’ in (ii), (iii) and (iv) follow by similar lines of reasoning. The implications ‘ \Leftarrow ’ in (i)–(iv) follow from (2.5) in Proposition 2.4. \square

Examples. We give next a number of examples of g -deviation measures.

Example 2.7 The family of g -deviation measures with driver functions given by

$$g_{c,d}(t, h, \tilde{h}) = c|h| + d \sqrt{\int_{\mathbb{R}^k \setminus \{0\}} |\tilde{h}(x)|^2 \nu(dx)}, \quad c, d \in \mathbb{R}_+ \setminus \{0\}, \quad (2.7)$$

corresponds to a measurement of the risk of a random variable $X \in L^2(\mathcal{F}_T)$ by the integrated multiples of the local volatilities of the continuous and discontinuous martingale parts in its martingale representation (2.1).

Example 2.8 In the case of a g -deviation measure with driver function given by

$$g(\omega, t, h, \tilde{h}) = CVaR_{t,a}^\nu(\tilde{h}), \quad a \in (0, \nu(\mathbb{R}^k \setminus \{0\})),$$

the risk is measured in terms of the values of the (large) jump sizes under $CVaR_{t,a}^\nu$. Here $CVaR_{t,a}^\nu(\tilde{h}) = \frac{1}{a} \int_0^a VaR_{t,b}^\nu(\tilde{h}) db$ is given in terms of the left-quantiles $VaR_{t,a}^\nu(\tilde{h})$, $a \in (0, \nu(\mathbb{R}^k \setminus \{0\}))$ of $h(J)$ under the measure $\nu(dx)$, that is,

$$VaR_{t,a}^\nu(\tilde{h}) := VaR_a^\nu(h(J)) := \sup\{y \in \mathbb{R} : \nu(\{x \in \mathbb{R}^k \setminus \{0\} : \tilde{h}(x) < -y\}) < a\}.$$

In the next example we deploy the following auxiliary result:

Proposition 2.9 *Let $I := \{t_0, t_1, \dots, t_n\} \subset [0, T]$ be strictly ordered. $D = (D_t)_{t \in I}$ satisfies (D1)–(D4) and (D6) if and only if for some collection $\tilde{D} = (\tilde{D}_t)_{t \in I}$ of conditional deviation measures we have*

$$D_t(X) = \mathbb{E} \left[\sum_{t_i \in I: t_i \geq t} \tilde{D}_{t_i} \left(\mathbb{E}[X | \mathcal{F}_{t_{i+1}}] - \mathbb{E}[X | \mathcal{F}_{t_i}] \right) \middle| \mathcal{F}_t \right], \quad t \in I, \quad X \in L^2(\mathcal{F}_T). \quad (2.8)$$

In particular, a dynamic deviation measure D satisfies (2.8) with $\tilde{D}_{t_i} = D_{t_i}$, $t_i \in I$.

Proof. ‘ \Leftarrow ’: We will only show that D_t satisfies (D6), as it is clear that (D1)–(D4) are satisfied. Let $X \in L^2(\mathcal{F}_T)$ and note that as $\tilde{D}_t, t \in I$, satisfy (D1) and (D4) we have for any $s, t \in I$ with $s > t$ that $D_t(\mathbb{E}[X|\mathcal{F}_s]) = \sum_{t_i \in I: t \leq t_i < s} \mathbb{E}[\tilde{D}_{t_i}(\mathbb{E}[X|\mathcal{F}_{t_{i+1}}])|\mathcal{F}_t]$. Thus, we have that $D_t(X)$ is equal to

$$\sum_{t_i \in I: t \leq t_i < s} \mathbb{E}[\tilde{D}_{t_i}(\mathbb{E}[X|\mathcal{F}_{t_{i+1}}])|\mathcal{F}_t] + \sum_{t_i \in I: s \leq t_i} \mathbb{E}[\tilde{D}_{t_i}(\mathbb{E}[X|\mathcal{F}_{t_{i+1}}])|\mathcal{F}_t] = D_t(\mathbb{E}[X|\mathcal{F}_s]) + \mathbb{E}[D_s(X)|\mathcal{F}_t].$$

‘ \Rightarrow ’: For $X \in L^2(\mathcal{F}_T)$ and $t_{i-1} \in I, i \geq 1$, we have by (D6) and (D1)

$$\begin{aligned} D_{t_{i-1}}(X) &= D_{t_{i-1}}(\mathbb{E}[X|\mathcal{F}_{t_i}]) + \mathbb{E}[D_{t_i}(X)|\mathcal{F}_{t_{i-1}}] \\ &= D_{t_{i-1}}(\mathbb{E}[X|\mathcal{F}_{t_i}] - \mathbb{E}[X|\mathcal{F}_{t_{i-1}}]) + \mathbb{E}[D_{t_i}(X)|\mathcal{F}_{t_{i-1}}]. \end{aligned} \quad (2.9)$$

An induction argument based on (2.9) then yields that (2.8) holds with $\tilde{D}_t = D_t, t \in I$. \square

Example 2.10 The formula (2.8) in Proposition 2.9 gives a way to define a collection $D = (D_t)_{t \in I}$ satisfying axioms (D1)–(D6) for $s, t \in I$, which we call a *dynamic deviation measure on the grid I*. Comparison of (2.8) and (2.5) suggests that one may obtain the values of a dynamic deviation measure as limit of the values of (suitably chosen) dynamic deviation measures on grids with vanishing mesh sizes. We next illustrate this for the g -deviation measures $\bar{D}^\lambda := D^{g^\lambda}, \lambda > 0$, corresponding to the driver functions g_λ given by

$$g_\lambda(\omega, t, h, \tilde{h}) := \lambda \sqrt{|h|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{h}(x)|^2 \nu(dx)}, \quad \lambda > 0, \quad (2.10)$$

and random variables $X \in L^2(\mathcal{F}_T)$ of the form

$$X = x + \int_0^T f(t) dW_t + \int_{[0, T] \times \mathbb{R}^k \setminus \{0\}} g(t, y) \tilde{N}(dt \times dy) \quad (2.11)$$

with $x \in \mathbb{R}, f \in C([0, T], \mathbb{R}^d)$ and $g \in C_0([0, T] \times \mathbb{R}^k, \mathbb{R})^\ddagger$. We construct approximating sequences in terms of the conditional *CVaR*-deviation measures given by $\tilde{D}_t(Y) := CVaR_{t, \alpha}(Y - \mathbb{E}[Y|\mathcal{F}_t])$ for $Y \in L^2(\mathcal{F}_T), t \in [0, T], \alpha \in (0, 1)$, where for $Z \in L^2(\mathcal{F}_T)$

$$CVaR_{t, \alpha}(Z) = \frac{1}{\alpha} \int_0^\alpha VaR_{t, b}(Z) db, \quad VaR_{t, b}(Z) = \sup\{y \in \mathbb{R} : \mathbb{P}(Z < -y|\mathcal{F}_t) < b\},$$

see Rockafellar *et al.* (2006a).

Specifically, the expression in (2.8) suggests to *scale* the value of conditional deviation measures corresponding to small time units in order to obtain in the limit a dynamic deviation measure. Denoting for X of the form (2.11)

$$M_{t_{i+1}} := \mathbb{E}[X|\mathcal{F}_{t_{i+1}}], \quad \Delta M_{i+1} := M_{t_{i+1}} - M_{t_i}, \quad t_i = Ti/2^n, \quad i = 0, \dots, 2^n - 1,$$

with $t_{2^n} = T$ and following this suggestion we specify the contribution to the total risk of

$$\Delta M_{i+1} = \int_{t_i}^{t_{i+1}} f(s) dW_s + \int_{(t_i, t_{i+1}] \times (\mathbb{R}^k \setminus \{0\})} g(s, y) \tilde{N}(ds \times dy), \quad i = 0, \dots, 2^n - 1,$$

$^\ddagger C([0, T], \mathbb{R}^d)$ and $C_0([0, T] \times \mathbb{R}^k, \mathbb{R})$ denote the sets of continuous functions $f : [0, T] \mapsto \mathbb{R}^d$, and of continuous functions $g : [0, T] \times \mathbb{R}^k \mapsto \mathbb{R}$ that are such that $\sup_{t \in [0, T]} |g(t, x)| \rightarrow 0$ as $|x| \rightarrow \infty$ and $\sup_{x \in \mathbb{R}^k \setminus \{0\}} \sup_{t \in [0, T]} \{|g(t, x)|/|x|^2\} < \infty$.

by $\tilde{D}_{t_i}(\Delta M_{i+1}) := \sqrt{\Delta t_{i+1}} CVaR_{t_i, \alpha}(\Delta M_{i+1})$, $\Delta t_{i+1} = t_{i+1} - t_i$, which gives rise to the dynamic deviation measure $D^{(n)} = (D_t^{(n)})_{t \in I_n}$ on $I_n := \{t_i, i = 0, \dots, 2^n\}$ given by

$$D_t^{(n)}(X) = \sum_{t_i \geq t} \mathbb{E} \left[\tilde{D}_{t_i}(\Delta M_{i+1}) \middle| \mathcal{F}_t \right] = \sum_{t_i \geq t} \sqrt{\sigma^2(t_i) \Delta t_{i+1}} \mathbb{E} \left[CVaR_{t_i, \alpha} \left(\frac{\Delta M_{i+1}}{\sqrt{\sigma^2(t_i) \Delta t_{i+1}}} \right) \middle| \mathcal{F}_t \right],$$

with $\sigma^2(t) := |f(t)|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |g(t, x)|^2 \nu(dx)$, $t \in I_n$, (2.12)

where we used that $CVaR_{t_i, \alpha}$ is positively homogeneous. As ΔM_{i+1} is infinitely divisible and f and g are bounded, we have by an application of Lindeberg-Feller Central Limit Theorem (see *e.g.*, Durrett (2004), p.129) that, when we let $n \rightarrow \infty$ while keeping t_i fixed the ratio $\Delta M_{i+1} / \sqrt{\sigma^2(t_i) \Delta t_{i+1}}$ converges in distribution to a standard normal random variable ξ . By uniform integrability and the independence of ΔM_{i+1} from \mathcal{F}_{t_i} we have

$$CVaR_{\alpha, t_i} \left(\frac{\Delta M_{i+1}}{\sqrt{\sigma^2(t_i) \Delta t_{i+1}}} \right) = CVaR_{\alpha} \left(\frac{\Delta M_{i+1}}{\sqrt{\sigma^2(t_i) \Delta t_{i+1}}} \right) \rightarrow CVaR_{\alpha}(\xi) = \frac{1}{\alpha} \int_0^{\alpha} \Phi^{-1}(u) du =: c_{\alpha},$$

where $CVaR_{\alpha}(\cdot) = CVaR_{\alpha, 0}(\cdot)$ and Φ^{-1} denotes the inverse of the standard normal distribution function Φ . Hence, letting $n \rightarrow \infty$ in (2.12) and deploying the uniform continuity of f and g we have for any $t \in [0, T]$ of the form $t = k/2^m$, $k, m \in \mathbb{N}$,

$$D_t^{(n)}(X) \rightarrow c_{\alpha} \mathbb{E} \left[\int_t^T \sqrt{|f(s)|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |g(s, x)|^2 \nu(dx)} ds \middle| \mathcal{F}_t \right] = \bar{D}_t^{c_{\alpha}}(X). \quad (2.13)$$

3 Characterisation theorem

We show next that any dynamic deviation measure that satisfies a domination condition is a g -deviation measure for some driver function g .

Definition 3.1 *A dynamic deviation measure $D = (D_t)_{t \in [0, T]}$ is called λ -dominated if for all $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$ we have*

$$D_t(X) \leq \bar{D}_t^{\lambda}(X).$$

Theorem 3.2 *Let $D = (D_t)_{t \in [0, T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \rightarrow L_+^2(\mathcal{F}_t)$, $t \in [0, T]$. Then D is a dynamic deviation measure that is λ -dominated for some $\lambda > 0$ if and only if there exists a convex and positively homogeneous driver function g of linear growth such that $D = D^g$. Furthermore, this driver function g is unique $d\mathbb{P} \times dt$ a.e.*

Proof. We first verify uniqueness: If \bar{g} is a driver function that satisfies $D^g = D^{\bar{g}}$, it follows from Proposition 2.6(iv) that $g = \bar{g}$ $d\mathbb{P} \times dt$ a.e. We note next that the implication ‘ \Leftarrow ’ follows from Proposition 2.4. The remainder is devoted to the proof of the implication ‘ \Rightarrow ’, which is established using a number of auxiliary results (the proofs of which are deferred to the end of the section).

Thus, let D be a given dynamic deviation measure that is λ -dominated, so that in particular D_0 is finite. We identify next a candidate driver function g . For the remainder of the proof we assume for the ease of presentation that $d = 1$. For fixed $h \in \mathbb{R}$ and $\tilde{h} \in L^2(\nu(dx))$ consider the mapping $\mu_{h, \tilde{h}} : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ given by

$$\mu_{h, \tilde{h}} : C_1 \times C_2 \mapsto D_0 \left((I_{C_1} h \cdot W)_T + (I_{C_2} \tilde{h} \cdot \tilde{N})_T \right).$$

Lemma 3.3 Let $(h, \tilde{h}) \in \mathbb{R} \times L^2(\nu(dx))$.

- (i) $C \mapsto \mu_{h, \tilde{h}}(C, \emptyset)$, $C \mapsto \mu_{h, \tilde{h}}(\emptyset, C)$ and $C \mapsto \mu_{h, \tilde{h}}(C, C)$ are σ -finite measures on $([0, T] \times \Omega, \mathcal{P})$.
- (ii) For any $C_1, C_2 \in \mathcal{P}$ we have

$$\mu_{h, \tilde{h}}(C_1, C_2) = \mu_{h, \tilde{h}}(C_1 \setminus C_2, \emptyset) + \mu_{h, \tilde{h}}(\emptyset, C_2 \setminus C_1) + \mu_{h, \tilde{h}}(C_1 \cap C_2, C_1 \cap C_2). \quad (3.1)$$

As D_0 is λ -dominated $C \mapsto \mu_{h, \tilde{h}}(C, C)$ is absolutely continuous with respect to the measure $d\mathbb{P} \times dt$ and we conclude from the Radon-Nikodym theorem that there exist an integrable non-negative density, say $R_{h, \tilde{h}}(s, \omega)$, that is such that $R_{0,0} = 0$ and for any set $C \in \mathcal{P}$

$$\mu_{h, \tilde{h}}(C, C) = \mathbb{E} \left[\int_0^T I_{C_s} R_{h, \tilde{h}}(s) ds \right], \quad (3.2)$$

where $C_s = \{\omega \in \Omega : (\omega, s) \in C\}$. In particular, we note that $\mu_{h, \tilde{h}}(C, \emptyset) = \mu_{h,0}(C, C)$ and $\mu_{h, \tilde{h}}(\emptyset, C) = \mu_{0, \tilde{h}}(C, C)$ satisfy (3.2) with $R_{h, \tilde{h}}$ replaced by $R_{0, \tilde{h}}$ and $R_{h,0}$ respectively. We define the candidate driver function g in terms of R by

$$g(t, \omega, h, \tilde{h}) := R_{h, \tilde{h}}(t, \omega), \quad (t, \omega) \in [0, T] \times \Omega. \quad (3.3)$$

The next result confirms that g is a driver function.

Lemma 3.4 *There exists a version of g such that, for $d\mathbb{P} \times dt$ a.e. (t, ω) , $(h, \tilde{h}) \mapsto g(t, \omega, h, \tilde{h})$ is continuous, convex, positively-homogeneous and dominated by g_λ .*

Note that $(t, \omega) \mapsto g(t, \omega, h, \tilde{h})$ is predictable for every $(h, \tilde{h}) \in \mathbb{R} \times L^2(\nu(dx))$ and by Lemma 3.4 $(h, \tilde{h}) \mapsto g(t, \omega, h, \tilde{h})$ is continuous in (h, \tilde{h}) , so that by standard arguments g can be approximated by $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{U}$ -measurable step functions and g itself may seem to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{U}$ -measurable. Note further that $g(t, \omega, h, \tilde{h})$ is non-negative as $R_{h, \tilde{h}}(t, \omega)$ is so for each (h, \tilde{h}) , and $g(s, \omega, 0, 0) = 0$ since the density $R_{0,0}(s, \omega)$ of the measure $\mu_{0,0}$ is zero. In the next result we show that D_0 may be identified with D_0^g .

Lemma 3.5 *Let g be as in Lemma 3.4. For $X \in L^2(\mathcal{F}_T)$ we have $D_0(X) = D_0^g(X)$.*

Lemma 3.5 and Remark 1.2(ii) imply that $D_t = D_t^g$ not only for $t = 0$ but also for all other $t \in (0, T]$. The proof is complete. \square

Proofs of Lemmas 3.3, 3.4 and 3.5

The proof of Lemma 3.3 is based on the following auxiliary result:

Proposition 3.6 *Let D be a dynamic deviation measure and $t \in [0, T]$. If $A_1, \dots, A_n \in \mathcal{F}_t$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ and $X_1, \dots, X_n \in L^2(\mathcal{F}_T)$, then for any $t \in [0, T]$*

$$D_t \left(\sum_{i=1}^n I_{A_i} X_i \right) = \sum_{i=1}^n D_t(I_{A_i} X_i). \quad (3.4)$$

Proof. Set $S_k := \sum_{i=1}^k I_{A_i} X_i$ and $B_k = \cup_{i=1}^k A_i$, $k = 1, \dots, n$. Let us first show by an induction argument that

$$D_t(S_n) = \sum_{i=1}^n I_{A_i} D_t(X_i). \quad (3.5)$$

Eqn. (3.4) is a direct consequence of (3.5) and (1.1). Using (1.1) and the fact $B_{n-1} \cap A_n = \emptyset$ we have

$$\begin{aligned} D_t(S_n) &= D_t(I_{B_{n-1}}S_{n-1} + I_{B_{n-1}^c}I_{A_n}X_n) = I_{B_{n-1}}D_t(S_{n-1}) + I_{B_{n-1}^c}D_t(I_{A_n}X_n) \\ &= I_{B_{n-1}}\sum_{i=1}^{n-1}I_{A_i}D_t(X_i) + I_{B_{n-1}^c}I_{A_n}D_t(X_n) = \sum_{i=1}^n I_{A_i}D_t(X_i), \end{aligned}$$

where we used (1.1) and the induction assumption in the third equality. This completes the proof of (3.5) and hence of the Lemma. \square

Proof of Lemma 3.3. (i) Let us first show that $C \mapsto \mu_{h,\tilde{h}}(C, \emptyset)$ constitutes a σ -finite measure. Clearly, $\mu_{h,\tilde{h}}(\cdot, \emptyset)$ is non-negative and $\mu_{h,\tilde{h}}(\emptyset, \emptyset) = 0$. Next we verify that $C \mapsto \mu_{h,\tilde{h}}(C, \emptyset)$ is additive for disjoint sets of the form $C_1 := (t_1, t_2] \times A$ and $C_2 := (t_3, t_4] \times B$ with $A \in \mathcal{F}_{t_1}$ and $B \in \mathcal{F}_{t_3}$. We consider first the case $t_1 \leq t_3 \leq t_2 \leq t_4$ and $A \cap B = \emptyset$ (note that in this case $C_1 \cap C_2 = \emptyset$). By deploying Propositions 2.9 and 3.6 we note that $\mu_{h,\tilde{h}}((t_1, t_2] \times A) \cup ((t_3, t_4] \times B), \emptyset)$ is equal to

$$\begin{aligned} &D_0(I_A h \cdot W)_{t_1, t_3} + (I_{A \cup B} h \cdot W)_{t_3, t_2} + (I_B h \cdot W)_{t_2, t_4} \\ &= \mathbb{E}[D_{t_1}((I_A h \cdot W)_{t_1, t_3})] + \mathbb{E}[D_{t_3}((I_{A \cup B} h \cdot W)_{t_3, t_2})] + \mathbb{E}[D_{t_2}((I_B h \cdot W)_{t_2, t_4})] \\ &= \mathbb{E}[D_{t_1}((I_A h \cdot W)_{t_1, t_3})] + \mathbb{E}[D_{t_3}((I_A h \cdot W)_{t_3, t_2})] \\ &\quad + \mathbb{E}[D_{t_3}((I_B h \cdot W)_{t_3, t_2})] + \mathbb{E}[D_{t_2}((I_B h \cdot W)_{t_2, t_4})] \\ &= D_0((I_A h \cdot W)_{t_1, t_2}) + D_0((I_B h \cdot W)_{t_3, t_4}), \end{aligned}$$

which is equal to $\mu_{h,\tilde{h}}((t_1, t_2] \times A, \emptyset) + \mu_{h,\tilde{h}}((t_3, t_4] \times B, \emptyset)$. The cases $t_1 \leq t_2 < t_3 \leq t_4$ and $t_1 \leq t_3 \leq t_4 \leq t_2$ may be verified in a similar manner. Thus, we may conclude that $\mu_{h,\tilde{h}}$ is additive on disjoint sets of the form $(t_1, t_2] \times A$ and $(t_3, t_4] \times B$. As D_0 is continuous in $L^2(\mathcal{F}_T)$ (see Remark 1.2(iii)) and the collection of sets considered above is a semi-algebra generating the predictable σ -algebra it follows that $\mu_{h,\tilde{h}}(\cdot, \emptyset)$ is σ -finite. The proofs that $C \mapsto \mu_{h,\tilde{h}}(\emptyset, C)$ and $C \mapsto \mu_{h,\tilde{h}}(C, C)$ are σ -finite measures are analogous, replacing in the equations above the term $h \cdot W$ by $\tilde{h} \cdot \tilde{N}$ and $(h \cdot W + \tilde{h} \cdot \tilde{N})$, respectively. **(ii)** Define C_1, C_2 as in (i) and consider the case $t_1 \leq t_3 \leq t_2 \leq t_4$ with general (not necessarily disjoint) $A \in \mathcal{F}_{t_1}$ and $B \in \mathcal{F}_{t_3}$. Expressing $X = I_A(h \cdot W)_{t_1, t_2} + I_B(\tilde{h} \cdot \tilde{N})_{t_3, t_4}$ as the sum of martingale increments

$$X = I_A(h \cdot W)_{t_1, t_3} + I_{A \setminus B}(h \cdot W)_{t_3, t_2} + I_{A \cap B}[(h \cdot W)_{t_3, t_2} + (\tilde{h} \cdot \tilde{N})_{t_3, t_2}] + I_{B \setminus A}(\tilde{h} \cdot \tilde{N})_{t_3, t_2} + I_B(\tilde{h} \cdot \tilde{N})_{t_2, t_4}$$

and using Propositions 2.9 and 3.6 we have that $\mu_{h,\tilde{h}}(C_1, C_2) = D_0(X)$ is equal to

$$\begin{aligned} &\mathbb{E}[D_{t_1}(I_A(h \cdot W)_{t_1, t_3})] + \mathbb{E}\left[D_{t_3}\left(I_{A \setminus B}(h \cdot W)_{t_3, t_2} + I_{A \cap B}[(h \cdot W)_{t_3, t_2} + (\tilde{h} \cdot \tilde{N})_{t_3, t_2}] + I_{B \setminus A}(\tilde{h} \cdot \tilde{N})_{t_3, t_2}\right)\right] \\ &\quad + \mathbb{E}\left[D_{t_2}(I_B(\tilde{h} \cdot \tilde{N})_{t_2, t_4})\right] \\ &= \mathbb{E}[D_{t_1}(I_A(h \cdot W)_{t_1, t_3})] + \mathbb{E}[D_{t_3}(I_{A \setminus B}(h \cdot W)_{t_3, t_2})] + \mathbb{E}\left[D_{t_3}(I_{A \cap B}[(h \cdot W)_{t_3, t_2} + (\tilde{h} \cdot \tilde{N})_{t_3, t_2}])\right] \\ &\quad + \mathbb{E}\left[D_{t_3}(I_{B \setminus A}(\tilde{h} \cdot \tilde{N})_{t_3, t_2})\right] + \mathbb{E}\left[D_{t_2}(I_B(\tilde{h} \cdot \tilde{N})_{t_2, t_4})\right]. \end{aligned}$$

Thus, using Proposition 2.9 again we have

$$\begin{aligned} \mu_{h,\tilde{h}}(C_1, C_2) &= D_0(I_A(h \cdot W)_{t_1, t_3} + I_{A \setminus B}(h \cdot W)_{t_3, t_2}) + D_0(I_{B \cap A}[(h \cdot W)_{t_3, t_2} + (\tilde{h} \cdot \tilde{N})_{t_3, t_2}]) \\ &\quad + D_0(I_{B \setminus A}(\tilde{h} \cdot \tilde{N})_{t_3, t_2} + I_B(\tilde{h} \cdot \tilde{N})_{t_2, t_4}) \\ &= \mu_{h,\tilde{h}}(C_1 \setminus C_2, \emptyset) + \mu_{h,\tilde{h}}(C_1 \cap C_2, C_1 \cap C_2) + \mu_{h,\tilde{h}}(\emptyset, C_2 \setminus C_1). \end{aligned}$$

The cases $t_1 \leq t_2 < t_3 \leq t_4$ and $t_1 \leq t_3 \leq t_4 \leq t_2$ may be verified in a similar manner. By the continuity of D_0 (Remark 1.2(iii)) and monotone class arguments (by keeping first C_1 and then C_2 fixed) it follows that (3.1) holds for all predictable sets, as asserted. \square

Proof of Lemma 3.4. First of all, note that the predictable σ -algebra is generated by countable many sets, say A_1, A_2, \dots . Fix $n \in \mathbb{N}$ and denote $\mathcal{P}^n := \sigma(A_1, \dots, A_n)$. By considering finer partitions we may after relabeling assume without loss of generality that the A_i are disjoint. Denote by η the measure $\eta := d\mathbb{P} \times dt$ on $(\Omega \times [0, T], \mathcal{P})$ and let $R_{h, \tilde{h}}^n = E_\eta[R_{h, \tilde{h}} | \mathcal{P}^n]$.[§] Since the filtration is generated by the disjoint sets A_1, A_2, \dots, A_n it is standard to note that

$$R_{h, \tilde{h}}^n(s, \omega) = \sum_{i: \nu(A_i) \neq 0} \frac{I_{A_i}(s, \omega)}{\eta(A_i)} \mu_{h, \tilde{h}}(A_i, A_i) \text{ for } d\mathbb{P} \times ds \text{ a.e. } (s, \omega). \quad (3.6)$$

By possibly modifying $R_{h, \tilde{h}}^n$ on a zero-set we may assume that (3.6) holds for all $(s, \omega) \in [0, T] \times \Omega$. It follows from (3.6) and the convexity and positive homogeneity of $(h, \tilde{h}) \rightarrow \mu_{h, \tilde{h}}(A_i, A_i)$ that, for all fixed (s, ω) , $R_{h, \tilde{h}}^n(s, \omega)$ is convex and positively homogeneous in (h, \tilde{h}) . Furthermore, we claim that $|R_{h, \tilde{h}}^n| \leq g_\lambda(h, \tilde{h})$. For suppose this were not the case, that is, for some (h, \tilde{h}) and A_i , $|R_{h, \tilde{h}}^n| > g_\lambda(h, \tilde{h})$ for all $(s, \omega) \in A_i$. Then we would have for $X = (H \cdot W)_T + (\tilde{H} \cdot \tilde{N})_T$ with $H_s = hI_{A_i}$ and $\tilde{H}_s = \tilde{h}I_{A_i}$ that $D_0(X) = \mu_{h, \tilde{h}}(A_i, A_i) = \mathbb{E} \left[\int_0^T I_{A_i}(s) R_{h, \tilde{h}}^n(s) ds \right]$ satisfies

$$D_0(X) > \mathbb{E} \left[\int_0^T I_{A_i}(s) g_\lambda(h, \tilde{h}) ds \right] = \mathbb{E} \left[\int_0^T g_\lambda(H_s, \tilde{H}_s) ds \right] = \bar{D}_0^\lambda(X),$$

which is in contradiction with the fact that D is λ -dominated.

Since \mathcal{P}^n is an increasing sequence of σ -algebras with $\cup_{n=1}^\infty \mathcal{P}^n = \mathcal{P}$ it follows from the martingale convergence theorem that $R_{h, \tilde{h}}^n(t, \omega) = E_\eta[R_{h, \tilde{h}} | \mathcal{P}^n](t, \omega)$ converges to $E_\eta[R_{h, \tilde{h}} | \mathcal{P}](t, \omega) = R_{h, \tilde{h}}(t, \omega)$ for $d\mathbb{P} \times dt$ a.e. (t, ω) . This convergence only holds up to a zero set. On this zero set, we may set $R_{h, \tilde{h}}(t, \omega)$ equal to $\limsup_n R_{h, \tilde{h}}^n(t, \omega)$. Hence, this version of $R_{h, \tilde{h}}$ is dominated by g_λ and is convex and positively homogeneous in (h, \tilde{h}) for every $(t, \omega) \in [0, T] \times \Omega$ as the limit of convex and positively homogeneous functions. The asserted continuity follows since every convex function that is locally bounded is continuous (see Theorem 2.2.9 in Zalinescu (2002)). \square

Proof of Lemma 3.5. We split the proof in four steps.

Step 1: For $X = ((hI_{C_1}) \cdot W)_T + ((\tilde{h}I_{C_2}) \cdot \tilde{N})_T$ for $(h, \tilde{h}) \in \mathbb{R} \times L^2(\nu(dx))$ and $C_1, C_2 \in \mathcal{P}$, we find by using $g(t, \omega, 0, 0) = 0$ that $D_0^g(X) = \mathbb{E} \left[\int_0^T g(s, hI_{C_1}(s), \tilde{h}I_{C_2}(s)) ds \right]$ is equal to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T I_{C_1 \setminus C_2}(s) g(s, h, 0) ds \right] + \mathbb{E} \left[\int_0^T I_{C_2 \setminus C_1}(s) g(s, 0, \tilde{h}) ds \right] + \mathbb{E} \left[\int_0^T I_{C_1 \cap C_2}(s) g(s, h, \tilde{h}) ds \right] \\ &= \mu_{h, \tilde{h}}(C_1 \setminus C_2, \emptyset) + \mu_{h, \tilde{h}}(\emptyset, C_2 \setminus C_1) + \mu_{h, \tilde{h}}(C_1 \cap C_2, C_1 \cap C_2), \end{aligned} \quad (3.7)$$

which is by (3.1) equal to $\mu_{h, \tilde{h}}(C_1, C_2) = D_0(X)$ (note that we only have to integrate over $C_1 \cup C_2$ as $g(t, \omega, 0, 0) = 0$).

Step 2: Fix $t_i, t_{i+1} \in [0, T]$ with $t_i < t_{i+1}$ and let $X = ((h_i I_{(t_i, t_{i+1}]}) \cdot W)_{t_i, t_{i+1}} + ((\tilde{h}_i I_{(t_i, t_{i+1}]}) \cdot \tilde{N})_{t_i, t_{i+1}}$ with $h_i := \sum_{j=1}^m c_j I_{A_j}$, $\tilde{h}_i := \sum_{j=1}^m \tilde{c}_j I_{A_j}$, and $c_j \in \mathbb{R}$, $\tilde{c}_j \in L^2(\nu(dx))$, and disjoint sets $A_j \in \mathcal{F}_{t_i}$,

[§]Specifically, $R_{h, \tilde{h}}^n$ is the \mathcal{P}^n -measurable random variable satisfying $E_\eta[R_{h, \tilde{h}} U] = E_\eta[R_{h, \tilde{h}}^n U]$ for all bounded \mathcal{P}^n random variables U , with $E_\eta[Z] = \int_0^T E[Z(s)] ds$ for $Z \in L^1(\eta)$.

$j = 1, \dots, m$, satisfying $\cup_j A_j = \Omega$ (we may assume w.l.o.g. that the A_j are the same for h and \tilde{h} by setting some c_j and \tilde{c}_j equal to zero). By step 1, denoting $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$,

$$\begin{aligned} \mathbb{E} \left[\int_{t_i}^{t_{i+1}} g(s, I_{A_j} c_j, I_{A_j} \tilde{c}_j) ds \right] &= D_0 \left(I_{A_j} c_j \Delta W_{i+1} + \int_{\mathbb{R}^k \setminus \{0\}} I_{A_j} \tilde{c}_j(x) \tilde{N}((t_i, t_{i+1}] \times dx) \right) \\ &= \mathbb{E} \left[D_{t_i} \left(I_{A_j} c_j \Delta W_{i+1} + \int_{\mathbb{R}^k \setminus \{0\}} I_{A_j} \tilde{c}_j(x) \tilde{N}((t_i, t_{i+1}] \times dx) \right) \right]. \end{aligned}$$

Hence by Proposition 2.9 $D_0(X)$ is equal to

$$\sum_{j=1}^m \mathbb{E} \left[D_{t_i} \left(I_{A_j} c_j \Delta W_{t_{i+1}} + \int_{\mathbb{R}^k \setminus \{0\}} I_{A_j} \tilde{c}_j(x) \tilde{N}((t_i, t_{i+1}] \times dx) \right) \right] = \mathbb{E} \left[\sum_{j=1}^m \int_{t_i}^{t_{i+1}} g(s, I_{A_j} c_j, I_{A_j} \tilde{c}_j) ds \right],$$

which is equal to $\mathbb{E} \left[\int_0^T g(s, h_s, \tilde{h}_s) ds \middle| \mathcal{F}_t \right] = D_0^g(X)$.

Step 3: Let $0 \leq t_1 < \dots < t_n = T$ be given. For simple functions $X = \left((\sum_{i=1}^l h_i I_{(t_i, t_{i+1}]}) \cdot W \right)_T + \left((\sum_{i=1}^l \tilde{h}_i I_{(t_i, t_{i+1}]}) \cdot \tilde{N} \right)_T$ for $l \in \mathbb{N}$, with h_i and \tilde{h}_i as in step 2 we have by Proposition 2.9, step 2 and Proposition 2.4

$$\begin{aligned} D_0(X) &= \sum_{i=1}^l \mathbb{E} \left[D_{t_i} \left(((h_i I_{(t_i, t_{i+1}]}) \cdot W)_{t_i, t_{i+1}} + ((\tilde{h}_i I_{(t_i, t_{i+1}]}) \cdot \tilde{N})_{t_i, t_{i+1}} \right) \right] \\ &= \sum_{i=1}^l \mathbb{E} \left[D_{t_i}^g \left(((h_i I_{(t_i, t_{i+1}]}) \cdot W)_{t_i, t_{i+1}} + ((\tilde{h}_i I_{(t_i, t_{i+1}]}) \cdot \tilde{N})_{t_i, t_{i+1}} \right) \right] = D_0^g(X). \end{aligned}$$

Hence, we have $D_0(X) = D_0^g(X)$ for all simple functions X .

Step 4: That $D_0(X) = D_0^g(X)$ not only for simple functions but also for general $X \in L^2(\mathcal{F}_T)$ follows by the continuity of D_0^g and D_0 in Lemma 2.5 (note that g is of linear growth) and Remark 1.2(iii). \square

4 Representation results, m -stability and time-consistency

We next turn to a dual representation result for general dynamic deviation measures which is, as we show in Theorem 4.4, given in terms of *additively m -stable* representing sets (see Definition 4.2). Specifically, we show that additive m -stability is in some sense necessary and sufficient to obtain the time-consistency axiom (D6)—see Proposition 4.9. The proof of these results rests on auxiliary dual representation results. Using these results we first establish in Theorem 4.1 that an integral representation of the form (2.5) holds for any dynamic deviation measure even if the domination condition is not satisfied.

In particular, we may strengthen the characterisation of dynamic deviation measures given in Theorem 3.2 as follows:

Theorem 4.1 *Let $D = (D_t)_{t \in [0, T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t)$, $t \in [0, T]$. Then D is a dynamic deviation measure if and only if there exists a convex positively homogeneous driver function g such that for any $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$*

$$D_t(X) = \mathbb{E} \left[\int_t^T g(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t \right] \quad (4.1)$$

and $\mathbb{E} \left[\int_0^T g(s, H_s^X, \tilde{H}_s^X)^2 ds \right] < \infty$.

The mentioned notion of additive m -stability is the requirement of stability under additive pasting of subsets of the collections of (conditionally) zero-mean random variables given by

$$\mathcal{Q}_{\mathcal{F}_t} := \{\xi \in L^2(\mathcal{F}_T) | \mathbb{E}[\xi | \mathcal{F}_t] = 0\}, \quad \mathcal{Q} := \mathcal{Q}_{\mathcal{F}_0} = \{\xi \in L^2(\mathcal{F}_T) | \mathbb{E}[\xi] = 0\}.$$

Definition 4.2 A set $\mathcal{S} \subset \mathcal{Q}$ is called *additively m -stable* if for any $\xi^1, \xi^2 \in \mathcal{S}$ and $t \in [0, T]$, $\xi^2 + \mathbb{E}[\xi^1 - \xi^2 | \mathcal{F}_t]$ defines an element of \mathcal{S} .

Denoting for a given set $\mathcal{S} \subset \mathcal{Q}$

$$\mathcal{S}_{s,t} := \{\mathbb{E}[\xi | \mathcal{F}_t] - \mathbb{E}[\xi | \mathcal{F}_s] | \xi \in \mathcal{S}\}, \quad s, t \in [0, T],$$

we note that $\mathcal{S} = \mathcal{S}_{0,T}$ and that a necessary and sufficient condition for \mathcal{S} to be additively m -stable is

$$\mathcal{S} = \mathcal{S}_{0,t} + \mathcal{S}_{t,T}, \quad \text{for any } t \in [0, T],$$

where $A + B$ denotes the direct sum of the sets A and B .[¶]

Theorem 4.3 Let $D = (D_t)_{t \in [0, T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t)$, $t \in [0, T]$, satisfying (D_4) . Then D is a dynamic deviation measure if and only if for some convex, bounded, closed subset \mathcal{S}^D of \mathcal{Q} that contains zero and is additively m -stable we have

$$D_t(X) = \text{ess sup}_{\xi \in \mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t}} \mathbb{E}[\xi X | \mathcal{F}_t], \quad t \in [0, T]. \quad (4.2)$$

In the next result we call a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable subset $C = (C_t)_{t \in [0, T]}$ of $[0, T] \times \Omega \times \mathbb{R}^d \times L^2(\nu(dx))$ closed, convex or non-empty if for $d\mathbb{P} \times dt$ a.e. $(t, \omega) \in [0, T] \times \Omega$, the sets $C_t(\omega)$ are closed, convex or non-empty. and we denote by $\text{int}(C)$ the collection of interiors of the sets $C_t(\omega)$, $(t, \omega) \in [0, T] \times \Omega$.

Theorem 4.4 Let $D = (D_t)_{t \in [0, T]}$ be a collection of maps $D_t : L^2(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t)$, $t \in [0, T]$. Then D is a dynamic deviation measure if and only if there exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable set $C^D = (C_t^D)_{t \in [0, T]}$ that is convex, closed with $0 \in \text{int}(C)$, such that D satisfies the representation in (4.2) with a bounded set \mathcal{S}^D given in terms of C^D by

$$\mathcal{S}^D = \left\{ \xi \in \mathcal{Q} \mid (H_t^\xi, \tilde{H}_t^\xi) \in C_t^D \text{ for all } t \in [0, T] \right\}. \quad (4.3)$$

The proofs of Theorems 4.1, 4.3 and 4.4 are given below.

Remark 4.5 (Relation to strong time-consistency of dynamic risk-measures) The characterisation in Theorem 4.4 is reminiscent of analogous characterisation results of (strong) time-consistency of dynamic risk measures available in the literature. If we call a set $\mathcal{S}' \subset \mathcal{M}$ *multiplicatively m -stable* if for every $\xi^1, \xi^2 \in \mathcal{S}'$ and $t \in [0, T]$ the element $L_t := \xi_t^1 \xi_T^2 / \xi_t^2$ is contained in \mathcal{S}' , we note that under multiplicative m -stability of \mathcal{S}' we have the decomposition $\mathcal{S}' = \mathcal{S}'_{0,T} = \mathcal{S}'_{0,t} \mathcal{S}'_{t,T}$ with $\mathcal{S}'_{s,t} := \{\mathbb{E}[\xi | \mathcal{F}_t] / \mathbb{E}[\xi | \mathcal{F}_s] | \xi \in \mathcal{S}'\}$ (with $0/0 = 0$), so that the set \mathcal{S}' is stable under ‘multiplicative’ pasting. It is well-known that coherent risk measures are (strongly) time-consistent precisely if the representing sets in the corresponding dual representations are multiplicatively m -stable; see among many others Chen and Epstein (2002) (where multiplicative m -stability is called ‘rectangular property’), Riedel (2004), Delbaen (2006), Artzner *et al.* (2007) or Föllmer and Schied (2011).

[¶]That is, $A + B := \{a + b : a \in A, b \in B\}$

Specifically, in a Brownian setting it is shown in Delbaen (2006) that multiplicative m -stability of a convex and closed set $\mathcal{S}' \subset \mathcal{M} := \{\xi \in L_+^1(\mathcal{F}_T) | \mathbb{E}[\xi] = 1\}$ containing 1 corresponds to the existence of a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable, closed and convex set C' containing 0 such that $\mathcal{S}' = \{\xi \in \mathcal{M} | (q_s^\xi, \psi_s^\xi) \in C'_s \text{ for all } s \in [0, T]\}$, where (q^ξ, ψ^ξ) is related to the stochastic logarithm of ξ by $\xi = \mathcal{E}\left((q^\xi \cdot W)_T + (\psi^\xi \cdot \tilde{N})_T\right)$ with $\mathcal{E}(\cdot)$ denoting the Doléans-Dade exponential. This result implies that time-consistent coherent risk measures on L^∞ satisfy the representation

$$\begin{aligned} \rho_t(X) &= \text{ess sup}_{\xi \in \mathcal{S}' \cap \mathcal{M}_{\mathcal{F}_t}} \mathbb{E}[-\xi X | \mathcal{F}_t], \text{ with } \mathcal{M}_{\mathcal{F}_t} := \{\xi \in L_+^1(\mathcal{F}_T) | \mathbb{E}[\xi | \mathcal{F}_t] = 1\} \text{ and} \\ \mathcal{S}' &= \left\{ \xi \in L_+^1(\mathcal{F}_T) | (q_s^\xi, \psi_s^\xi) \in C'_s \text{ for all } s \in [0, T] \right\}. \end{aligned} \quad (4.4)$$

This result is generalized in Delbaen *et al.* (2010) to convex risk measures. As a counterpart of Theorem 3.1 in Delbaen (2006), which concerns multiplicatively m -stable sets in a Brownian filtration, we have from Theorem 4.4 and Propositions 4.6–4.9 below that *a closed and convex set $\mathcal{S} \subset \mathcal{Q}$ containing 0 is additively m -stable if and only if, for some $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable set $C^* = (C_t^*)_{t \in [0, T]}$ that is convex, closed and contains 0, we have $\mathcal{S} = \{\xi \in \mathcal{Q} | (H_t^\xi, \tilde{H}_t^\xi) \in C_t^* \text{ for all } t \in [0, T]\}$.*

Auxiliary representation results. Our starting point is the \mathcal{F}_t -conditional version of the duality result given in Theorem 1 in Rockafellar *et al.* (2006a).

Proposition 4.6 *Let $t \in [0, T]$ and let the map $D_t : L^2(\mathcal{F}_T) \rightarrow L^0(\mathcal{F}_t)$ be given.*

(i) *D_t satisfies (D1)–(D3) and (D5) and maps $L^2(\mathcal{F}_T)$ to $L_+^2(\mathcal{F}_t)$ if and only if there exists a bounded, closed and convex set $\mathcal{S}_{D_t} \subset \mathcal{Q}_{\mathcal{F}_t}$ containing zero such that*

$$D_t(X) = \text{ess sup}_{\xi \in \mathcal{S}_{D_t}} \mathbb{E}[\xi X | \mathcal{F}_t], \quad X \in L^2(\mathcal{F}_T). \quad (4.5)$$

The set \mathcal{S}_{D_t} is uniquely determined by its (convex) indicator function $J_{\mathcal{S}_{D_t}} : L^2(\mathcal{F}_T) \rightarrow \{0, \infty\}$ given by

$$J_{\mathcal{S}_{D_t}}(\xi) := \text{ess sup}_{X \in L^2(\mathcal{F}_T)} \{\mathbb{E}[\xi X | \mathcal{F}_t] - D_t(X)\}. \quad (4.6)$$

(ii) *Assume the conditions in (i) are satisfied. Then D_t satisfies (D4) if and only if for every $X \in L^2(\mathcal{F}_T)$ with $X \notin L^2(\mathcal{F}_t)$ there exists $\xi \in \mathcal{S}_{D_t}$ such that $\mathbb{P}[\mathbb{E}[\xi X | \mathcal{F}_t] > 0] > 0$.*

Remark 4.7 Note that by (4.6) we have for any set $A \in \mathcal{F}_t$ and $\xi^1, \xi^2 \in \mathcal{S}_{D_t}$ that $I_A \xi^1 + I_{A^c} \xi^2 \in \mathcal{S}_{D_t}$. Sets having this property are *directed*.^{||}

Hence, $D_t(X)$ admits a robust representation with representing set given by a collection of signed measures. This proposition is stated in Rockafellar *et al.* (2006a) in a static setting but it can be seen to also hold true conditionally on \mathcal{F}_t —see for instance Riedel (2004), Ruszczyński and Shapiro (2006), or Cheridito and Kupper (2011) for related arguments.

For dynamic deviation measures the property (D6) induces a specific structure of the sets \mathcal{S}_{D_t} , $t \in [0, T]$, which we specify in the next results. A first observation is as follows:

Proposition 4.8 *Let $t \in [0, T]$ and let D be a dynamic deviation measure and denote $\mathcal{S}^D := \mathcal{S}_{D_T}$. We have that the set \mathcal{S}_{D_t} in the representation (4.5) of D_t is such that $\mathcal{S}_{D_t} = \mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t} = \mathcal{S}_{t,T}^D$.*

Proof of Proposition 4.8. Let $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$. For brevity we denote throughout the proof $\mathcal{S} = \mathcal{S}^D$, $\mathcal{S}_t = \mathcal{S}_{D_t}$ and $\mathcal{S}_{t,T} = \mathcal{S}_{t,T}^D$. As it is clear that $\mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t} = \mathcal{S}_{t,T}$ (noting that $\mathcal{S}_{t,T} \subset \mathcal{Q}_{\mathcal{F}_t}$), the remainder of the proof is concerned with showing that the sets $\mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$ and \mathcal{S}_t are equal.

^{||} A set \mathcal{S} is called directed if for any $\xi^1, \xi^2 \in \mathcal{S}$ there exists $\bar{\xi} \in \mathcal{S}$ with $\bar{\xi} \geq \xi^1 \vee \xi^2$.

Noting that $\mathbb{E}[D_t(X)] \leq D_0(X)$ (by (D6)), recalling (4.6) and deploying (D6), (D1) and the fact that $L^2(\mathcal{F}_T)$ is directed, we have for $\xi \in \mathcal{S}_t \subset \mathcal{Q}_{\mathcal{F}_t}$

$$\begin{aligned} J_{\mathcal{S}}(\xi) &= \sup_{X \in L^2(\mathcal{F}_T)} \{\mathbb{E}[\xi X] - D_0(X)\} \leq \sup_{X \in L^2(\mathcal{F}_T)} \{\mathbb{E}[\xi X] - \mathbb{E}[D_t(X)]\} \\ &= \sup_{X \in L^2(\mathcal{F}_T)} \mathbb{E}[\mathbb{E}[\xi X | \mathcal{F}_t] - D_t(X)] = \mathbb{E} \left[\text{ess sup}_{X \in L^2(\mathcal{F}_T)} \{\mathbb{E}[\xi X | \mathcal{F}_t] - D_t(X)\} \right] = 0, \end{aligned}$$

where in the last equality we used (4.6). As $J_{\mathcal{S}}(\xi)$ is either zero or infinity it follows from the previous display that $J_{\mathcal{S}}(\xi) = 0$ implying that $\xi \in \mathcal{S}$ and thus $\xi \in \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$. This shows $\mathcal{S}_t \subset \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$.

On the other hand, if $\xi \in \mathcal{S}_t^c := L^2(\mathcal{F}_T) \setminus \mathcal{S}_t$ then we have either (a) $\xi \in (L^2(\mathcal{F}_T) \setminus \mathcal{Q}_{\mathcal{F}_t}) \cap \mathcal{S}_t^c$ or (b) $\xi \in \mathcal{Q}_{\mathcal{F}_t} \cap \mathcal{S}_t^c$. In case (a) we have $\xi \notin \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$, while in case (b) (4.6) in Proposition 4.6 yields that there exists $X' \in L^2(\mathcal{F}_T)$ such that $\mathbb{E}[\xi X' | \mathcal{F}_t] - D_t(X') > 0$ on a non-zero set, say A . Hence by using (D6) and that $\xi \in \mathcal{Q}_{\mathcal{F}_t}$ we have (from (4.6) with $t = 0$)

$$\begin{aligned} J_{\mathcal{S}}(\xi) &\geq \mathbb{E}[\xi I_A X'] - \mathbb{E}[D_t(X' I_A)] \\ &= \mathbb{E}[I_A(\xi X' - D_t(X'))] = \mathbb{E}[I_A(\mathbb{E}[\xi X' | \mathcal{F}_t] - D_t(X'))] > 0. \end{aligned}$$

Thus, $J_{\mathcal{S}}(\xi) = \infty$ and we have that $\xi \notin \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$, also in case (b). Hence, $\mathcal{S}_t^c \subset L^2(\mathcal{F}_T) \setminus (\mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t})$. Combined with the inclusion derived in previous paragraph this yields that $\mathcal{S}_t = \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}$. \square

The following result shows that stability under ‘additive pasting’ of the representing set in the form of additive m -stability is a necessary and sufficient condition for (D6) to hold.

Proposition 4.9 *Let $\mathcal{S} \subset \mathcal{Q}$ be a convex, closed set containing zero. \mathcal{S} is additively m -stable if and only if the collection $D_t(X) := \text{ess sup}_{\xi \in \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}} \mathbb{E}[\xi X | \mathcal{F}_t]$, $t \in [0, T]$, $X \in L^2(\mathcal{F}_T)$, satisfies (D6).*

Proof. We first show ‘ \Rightarrow ’. We only give the proof that (D6) holds for $s = 0$ as the proof for $s \in (0, T]$ is analogous. Let $X \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$. Denoting $\xi_t = \mathbb{E}[\xi | \mathcal{F}_t]$ and $\xi_{t,T} = \xi - \xi_t$ for $\xi \in L^2(\mathcal{F}_T)$ we have

$$\begin{aligned} D_0(X) &= \sup_{\xi \in \mathcal{S}} \mathbb{E}[\xi X] = \sup_{\xi \in \mathcal{S}} \mathbb{E}[\mathbb{E}[\xi_t X + (\xi - \xi_t)X | \mathcal{F}_t]] \\ &= \sup_{\xi = \xi_t + \xi_{t,T} \in \mathcal{S}_{0,t} + \mathcal{S}_{t,T}} \{\mathbb{E}[\xi_t X] + \mathbb{E}[\mathbb{E}[\xi_{t,T} X | \mathcal{F}_t]]\} = \sup_{\xi_t \in \mathcal{S}_{0,t}, \xi_{t,T} \in \mathcal{S}_{t,T}} \{\mathbb{E}[\xi_t X] + \mathbb{E}[\mathbb{E}[\xi_{t,T} X | \mathcal{F}_t]]\}. \end{aligned}$$

Hence by the directedness of $\mathcal{S}_{t,T}$ (Remark 4.7) and Proposition 4.8 we obtain

$$\begin{aligned} D_0(X) &= \sup_{\xi_t \in \mathcal{S}_{0,t}} \mathbb{E}[\xi_t \mathbb{E}[X | \mathcal{F}_t]] + \sup_{\xi_{t,T} \in \mathcal{S}_{t,T}} \mathbb{E}[\mathbb{E}[\xi_{t,T} X | \mathcal{F}_t]] \\ &= \sup_{\xi \in \mathcal{S}} \mathbb{E}[\xi \mathbb{E}[X | \mathcal{F}_t]] + \mathbb{E} \left[\text{ess sup}_{\xi_{t,T} \in \mathcal{S}_{t,T}} \mathbb{E}[\xi_{t,T} X | \mathcal{F}_t] \right] \\ &= D_0(\mathbb{E}[X | \mathcal{F}_t]) + \mathbb{E} \left[\text{ess sup}_{\xi \in \mathcal{S} \cap \mathcal{Q}_{\mathcal{F}_t}} \mathbb{E}[\xi X | \mathcal{F}_t] \right] = D_0(\mathbb{E}[X | \mathcal{F}_t]) + \mathbb{E}[D_t(X)]. \end{aligned}$$

To see that we have ‘ \Leftarrow ’ suppose that $\xi^1, \xi^2 \in \mathcal{S}$ such that $\xi_t^1 + (\xi^2 - \xi_t^2) \notin \mathcal{S}$ for some $t \in [0, T]$. Then by the Hahn-Banach Theorem there exists a random variable $X \in L^2(\mathcal{F}_T)$ such that we have

$$E := \mathbb{E}[(\xi_t^1 + (\xi^2 - \xi_t^2))X] > \sup_{\xi \in \mathcal{S}} \mathbb{E}[\xi X] = D_0(X). \quad (4.7)$$

Using Proposition 4.8 we note $E = \mathbb{E}[\xi_t^1 \mathbb{E}[X | \mathcal{F}_t]] + \mathbb{E}[\mathbb{E}[(\xi^2 - \xi_t^2)X | \mathcal{F}_t]]$ may be bounded above by

$$D_0(\mathbb{E}[X | \mathcal{F}_t]) + \mathbb{E} \left[\text{ess sup}_{\xi \in \mathcal{S}_{t,T}} \mathbb{E}[\xi X | \mathcal{F}_t] \right] = D_0(\mathbb{E}[X | \mathcal{F}_t]) + \mathbb{E}[D_t(X)] = D_0(X).$$

This bound is a contradiction to (4.7), which proves the implication ‘ \Leftarrow ’.

Proof of Theorem 4.3. The assertion follows by combining Propositions 4.6, 4.8 and 4.9. \square

In the proofs of Theorems 4.1 and 4.9 we deploy, for a given dynamic deviation measure D , the sequence $(D^{(n)})_{n \in \mathbb{N}}$ of dynamic deviation measures $D^{(n)} = (D_t^{(n)})_{t \in [0, T]}$, $D_t^{(n)} : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t)$ defined by

$$D_t^{(n)}(X) := \operatorname{ess\,sup}_{\xi \in (\mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t}) \cap \mathcal{A}^n} \mathbb{E}[\xi X | \mathcal{F}_t], \text{ with} \quad (4.8)$$

$$\mathcal{A}^n := \left\{ \xi \in L^2(\mathcal{F}_T) \left| \sup_{s \in [0, T]} \left\{ |H_s^\xi|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{H}_s^\xi(x)|^2 \nu(dx) \right\} \leq n^2 \right. \right\}. \quad (4.9)$$

Lemma 4.10 *Let $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$ and, for a given dynamic deviation measure D , let $(D^{(n)})_{n \in \mathbb{N}}$ and $(\mathcal{A}^n)_{n \in \mathbb{N}}$ be as in (4.8)–(4.9).*

(i) *for any $n \in \mathbb{N}$, we have $D_t^{(n)}(X) \leq D_t^{(n+1)}(X)$ and $\mathcal{A}^{n+1} = \frac{n+1}{n} \mathcal{A}^n$. Moreover, $D_t^{(n)}(X) \nearrow D_t(X)$ in $L^2(\mathcal{F}_t)$ as $n \rightarrow \infty$.*

(ii) *for any $n \in \mathbb{N}$, $\mathcal{S} \cap \mathcal{A}^n$ contains zero and is closed, bounded, convex and additively m -stable.*

(iii) *For any $n \in \mathbb{N}$, $D^{(n)}$ is a dynamic deviation measure that is n -dominated.*

Proof. (i) It is easily verified that $\mathcal{A}^{n+1} = \frac{n+1}{n} \mathcal{A}^n$ so that $\mathcal{A}^n \subset \mathcal{A}^{n+1}$ for $n \in \mathbb{N}$. Hence, by (4.8) we have $D_t^{(n)}(X) \leq D_t^{(n+1)}(X)$ for $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T)$. Furthermore, as $(\mathcal{A}^n)_{n \in \mathbb{N}}$ is dense in $L^2(\mathcal{F}_T)$ and the set \mathcal{S}^D in Theorem 4.3 is bounded, we have that $D_t^{(n)}(X) \nearrow D_t(X)$ as $n \rightarrow \infty$.

(ii) Let $n \in \mathbb{N}$. It is straightforward to verify that \mathcal{A}^n contains zero and is closed, bounded and convex. Let us show next that \mathcal{A}^n is additively m -stable. Let $t \in [0, T]$ and $\xi^1, \xi^2 \in \mathcal{A}^n$ and denote $L = \xi^2 + \mathbb{E}[\xi^1 - \xi^2 | \mathcal{F}_t]$. Then the representing pair (H^L, \tilde{H}^L) of $L \in L^2(\mathcal{F}_T)$ is expressed in terms of the representing pairs (H^i, \tilde{H}^i) , $i = 1, 2$, of ξ^1, ξ^2 by $H_s^L = H_s^1 I_{[0, t]}(s) + H_s^2 I_{(t, T]}(s)$ and $\tilde{H}_s^L = \tilde{H}_s^1 I_{[0, t]}(s) + \tilde{H}_s^2 I_{(t, T]}(s)$. In particular, we have $\sup_{s \in [0, T]} \{ |H_s^L|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{H}_s^L(x)|^2 \nu(dx) \} \leq n^2$ so that $L \in \mathcal{A}^n$. Thus, \mathcal{A}^n is additively m -stable. Since the set \mathcal{S}^D is also closed, convex and additively m -stable, the same holds for $\mathcal{A}^n \cap \mathcal{S}$.

(iii) Let $n \in \mathbb{N}$. From Proposition 4.6 and part (ii) we conclude that $D^{(n)}$ satisfies (D1)–(D3) and (D5). Furthermore, from Proposition 4.9 and part (ii) we have that $D^{(n)}$ satisfies (D6). Let us show next that $D^{(n)}$ satisfies positivity (D4). Let $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T) \setminus L^2(\mathcal{F}_t)$. By Propositions 4.6 and 4.8 there exists a $\tilde{\xi} \in \mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t}$ such that $\mathbb{E}[\tilde{\xi} X | \mathcal{F}_t] > 0$ on a non-zero set. As $(\mathcal{A}^n)_{n \in \mathbb{N}}$ is increasing and dense in $L^2(\mathcal{F}_T)$ (as noted in the proof of part (i)), we can find a sequence $(\xi^m)_m$ such that $\xi^m \in \mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t} \cap \mathcal{A}^m$ converges to $\tilde{\xi}$ in $L^2(\mathcal{F}_T)$ as $m \rightarrow \infty$. Next, choose m' sufficiently large such that on a non-zero set, say A , we have $\mathbb{E}[\xi^{m'} X | \mathcal{F}_t] > 0$ (which is possible since $\xi^m X$ converges to $\tilde{\xi} X$ in L^1 as $m \rightarrow \infty$). Define $\xi^* \in \mathcal{S}^D \cap \mathcal{Q}_{\mathcal{F}_t} \cap \mathcal{A}^n$ by $\xi^* := \frac{n}{m'} \xi^{m'}$. Since on A we have $\mathbb{E}[\xi^* X | \mathcal{F}_t] = \frac{n}{m'} \mathbb{E}[\xi^{m'} X | \mathcal{F}_t] > 0$ we conclude from (4.8) that $D^{(n)}$ satisfies (D4).

Finally, by deploying the Cauchy-Schwarz inequality we note that $D_t^{(n)}(X)$ may be bounded above by

$$\begin{aligned} \sup_{\xi \in \mathcal{A}^n} \mathbb{E}[\xi X | \mathcal{F}_t] &= \sup_{\xi \in \mathcal{A}^n} \mathbb{E} \left[\int_0^T \left((H_s^\xi)^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s^\xi(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \middle| \mathcal{F}_t \right] \\ &\leq n \mathbb{E} \left[\int_0^T \sqrt{|H_s^X|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{H}_s^X(x)|^2 \nu(dx)} ds \middle| \mathcal{F}_t \right] = \bar{D}_t^n(X), \end{aligned} \quad (4.10)$$

where we denote by v^\top the transpose of the column vector $v \in \mathbb{R}^d$. \square

4.1 Proof of Theorem 4.1

With the previously established results in hand we can now complete the proof of Theorem 4.1. As the arguments in the proof of the implication ‘ \Leftarrow ’ in Theorem 3.2 carry over for the proof of ‘ \Rightarrow ’ in Theorem 4.1, the remainder of the proof is concerned with the proof of ‘ \Rightarrow ’. Let D be a dynamic deviation measure, $X \in L^2(\mathcal{F}_T)$ and denote by $(D^{(n)})_{n \in \mathbb{N}}$ the approximating sequence of dynamic deviation measures from Lemma 4.10. By Lemma 4.10(i,iii) and Theorem 3.2 the sequence $(D^{(n)}(X))_{n \in \mathbb{N}}$ is monotone increasing and there exists a sequence $(g^n)_{n \in \mathbb{N}}$ of convex and positively homogeneous driver functions such that (4.1) holds (with D and g replaced by $D^{(n)}$ and g^n). Therefore, by Proposition 2.6(iv), $g^n \leq g^{n+1}$ for $n \in \mathbb{N}$, so that we can define $g := \lim_{n \rightarrow \infty} g^n$. Clearly, g is convex, positively homogeneous and lower semi-continuous as the limit of functions having these properties. Furthermore, for $(h, \tilde{h}) \neq 0$ we have $g(\omega, t, h, \tilde{h}) \geq g^1(\omega, t, h, \tilde{h}) > 0$ and $g(\omega, t, 0, 0) = \lim_{n \rightarrow \infty} g^n(\omega, t, 0, 0) = 0$ $d\mathbb{P} \times dt$ a.e. Hence, g is a convex and positively homogeneous driver function. Finally, as $(g^n)_n$ is an increasing sequence of functions an application of the monotone convergence theorem yields

$$D_t(X) = \lim_n D_t^{(n)}(X) = \lim_n \mathbb{E} \left[\int_t^T g^n(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\int_t^T g(s, H_s^X, \tilde{H}_s^X) ds \middle| \mathcal{F}_t \right].$$

This completes the proof of Theorem 4.1.

4.2 Proof of Theorem 4.4

In the proof of Theorem 4.4 we deploy the following auxiliary result:

Lemma 4.11 (i) *Let g be a convex and positively homogeneous driver function and let the $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable set $C = (C_t)_{t \in [0, T]}$ be determined by*

$$J_{C_t}(u, \tilde{u}) = r(t, u, \tilde{u}) := \sup_{u \in \mathbb{R}^d, \tilde{u} \in L^2(\nu(dx))} \left\{ u^\top h + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{u}(x) \tilde{h}(x) \nu(dx) - g(t, u, \tilde{u}) \right\}$$

for $u \in \mathbb{R}^d$ and $\tilde{u} \in L^2(\nu(dx))$. Then $0 \in \text{int}(C_t)(\omega)$ $d\mathbb{P} \times dt$ a.e.

(ii) *Let $C^D = (C_t^D)_{t \in [0, T]}$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable set and let \mathcal{S}^D be given by the right-hand side of (4.3). If $0 \in \text{int}(C_t^D)(\omega)$ $d\mathbb{P} \times dt$ a.e. then, for any $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T) \setminus L^2(\mathcal{F}_t)$, there exists a $\xi' \in \mathcal{S}^D$ such that $\mathbb{P}(\mathbb{E}[\xi' X | \mathcal{F}_t] > 0) > 0$.*

Proof. To simplify notation we denote $z := (h, \tilde{h})$ and $y = (q, \psi)$ for elements $(h, \tilde{h}), (q, \psi)$ in the Hilbert space $\mathbb{R}^d \times L^2(\nu(dx))$. Further we denote $\langle y, z \rangle_* = qh + \int_{\mathbb{R}^k \setminus \{0\}} \psi(x) \tilde{h}(x) \nu(dx)$ and $|z|_* = \sqrt{|h|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{h}(x)|^2 \nu(dx)}$.

(i) Set $\mathcal{Z} := \{z \in \mathbb{R}^d \times L^2(\nu(dx)) \mid |z|_* = 1\}$ and for $z \in \mathcal{Z}$ and $\lambda \in \mathbb{R}$ we denote $z^\lambda := \lambda z$. By the positive homogeneity of g and the symmetry of the set \mathcal{Z} we have for fixed $y \in \mathbb{R}^d \times L^2(\nu(dx))$ that $r(t, y) = \sup_{z \in \mathcal{Z}, \lambda \in \mathbb{R}} \{\langle y, z^\lambda \rangle_* - g(t, z^\lambda)\}$ is equal to

$$r(t, y) = \sup_{z \in \mathcal{Z}, \lambda \geq 0} \left\{ \langle y, z^\lambda \rangle_* - g(t, z^\lambda) \right\} = \sup_{z \in \mathcal{Z}, \lambda \geq 0} \lambda \left\{ \langle y, z \rangle_* - g(t, z) \right\}. \quad (4.11)$$

The supremum in (4.11) is finite if and only if for all $z \in \mathcal{Z}$ $\langle y, z \rangle_* \leq g(t, z)$. Letting $(y_n)_n$ be a sequence such that $|y_n|_* \rightarrow 0$ and using the Cauchy-Schwarz inequality, we have that

$$\sup_{z \in \mathcal{Z}} |\langle y, z \rangle_*| \leq |y_n|_* \sup_{z \in \mathcal{Z}} |z|_* = |y_n|_* \rightarrow 0.$$

Since by assumption $g(t, z) > 0$ for every fixed $z \in \mathcal{Z}$ we have that from a certain n onwards $\langle y_n, z \rangle_* \leq g(t, z)$ so that $r(t, y_n) = 0$. As $r(t, \omega, y_n) = J_{C_t(\omega)}(y_n)$ this entails that $y_n \in C_t(\omega)$ from a certain n onwards for every sequence y_n that is such that $|y_n|_* \rightarrow 0$. Hence, $0 \in \text{int}(C_t(\omega))$.

(ii) Let $t \in [0, T]$ and $X \in L^2(\mathcal{F}_T) \setminus L^2(\mathcal{F}_t)$. For any $s \in [0, T]$, we note that if $0 \in \text{int}(C_s^D(\omega))$ then there exists $\varepsilon'_s(\omega) \in (0, 1]$ such that $|y|_* \leq \varepsilon'_s(\omega)$ implies $y \in C_s^D(\omega)$. Define $\lambda_s(\omega) := |(H_s^X(\omega), \tilde{H}_s^X(\omega))|_*^2$, $A = \{(s, \omega) \in [t, T] \times \Omega : \lambda_s(\omega) > 0\}$ and denote by $\varepsilon = (\varepsilon_s)_{s \in [0, T]}$ the process given by $\varepsilon_s(\omega) := I_A(s, \omega) \varepsilon'_s(\omega) / \lambda_s(\omega)$. Then $\xi' := (\varepsilon H^X \cdot W)_{t, T} + (\varepsilon \tilde{H}^X \cdot \tilde{N})_{t, T}$ is element of \mathcal{S}^D . Since $X \in L^2(\mathcal{F}_T) \setminus L^2(\mathcal{F}_t)$, the set A has positive $d\mathbb{P} \times dt$ -measure so that

$$\mathbb{E} [\mathbb{E} [X \xi' | \mathcal{F}_t]] = \mathbb{E} \left[\mathbb{E} \left[\int_t^T I_A \varepsilon'_s ds \middle| \mathcal{F}_t \right] \right] = \mathbb{E} \left[\int_t^T I_A \varepsilon'_s ds \right] > 0,$$

which implies, as $\mathbb{E} [X \xi' | \mathcal{F}_t]$ is nonnegative, that $\mathbb{P}(\mathbb{E} [X \xi' | \mathcal{F}_t] > 0) > 0$. \square

Proof of Theorem 4.4. Let us first show the implication ' \Leftarrow ': We note first that, as is straightforward to verify, \mathcal{S}^D given in (4.3) is additively m -stable, convex, bounded, closed and contains zero. Moreover, Lemma 4.11 and Proposition 4.6(ii) imply that, for any $t \in [0, T]$, $D_t : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t)$ defined by (4.2) satisfies (D4). Hence, by Theorem 4.3 $D = (D_t)_{t \in [0, T]}$ is a dynamic deviation measure.

We next turn to the proof of ' \Rightarrow '. In view of Theorem 4.3 it suffices to show that \mathcal{S}^D is given by the expression in (4.3). For any $n \in \mathbb{N}$ let $D^{(n)}$ be defined as in (4.8). As noted before $(D^{(n)})_{n \in \mathbb{N}}$ is a collection of dynamic deviation measures increasing to D (Lemma 4.10) and the corresponding sequence $(g^n)_{n \in \mathbb{N}}$ of driver functions is increasing and satisfies $g^n \leq g$ (Proposition 2.6(iv)), where g is the function in the representation (4.1) of D (in Theorem 4.1). For $u \in \mathbb{R}^d$, $\tilde{u} \in L^2(\nu(dx))$ and $n \in \mathbb{N}$ define

$$r^n(s, u, \tilde{u}) := \sup_{h \in \mathbb{Q}^d, \tilde{h} \in \{h_1, h_2, h_3, \dots\}} \left\{ u^\top h + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{u}(x) \tilde{h}(x) \nu(dx) - g^n(s, h, \tilde{h}) \right\},$$

where $\{h_1, h_2, h_3, \dots\}$ denotes a countable basis of $L^2(\nu(dx))$. Note that for any $n \in \mathbb{N}$ we have (i) r^n lower semi-continuous and convex in (u, \tilde{u}) and (ii) r^n is a (convex) indicator function of some convex and closed set, say $C^n = (C_s^n)_{s \in [0, T]}$. Furthermore, we note the following observations: (a) since r^n is the supremum of a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ measurable process and C^n is the set where r^n is equal to zero, we have that C^n is also $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ -measurable and (b) as the functions $g^n(s, h, \tilde{h})$ is continuous in (h, \tilde{h}) , r^n coincides with the dual conjugate of g^n , so that we have $d\mathbb{P} \times dt$ a.e.

$$g^n(s, \omega, h, \tilde{h}) = \sup_{(u, \tilde{u}) \in C_s^n(\omega)} \left\{ u^\top h + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{u}(x) \tilde{h}(x) \nu(dx) \right\}. \quad (4.12)$$

Moreover, we have that (c) as the sequence $(g^n)_n$ is increasing, $(r^n)_n$ is a decreasing sequence so that $C^n \subset C^{n+1}$ for any $n \in \mathbb{N}$. Denote $C = \cup_{n=1}^\infty C^n$ and note that C is convex and measurable as the increasing union of convex and measurable sets.

Let us next establish the representation (4.2) for $D^{(n)}(X)$ for given $n \in \mathbb{N}$ and $X \in L^2(\mathcal{F}_T)$. As $D^{(n)}(X) = D^{g^n}(X)$ we have

$$\begin{aligned} D_0^{(n)}(X) &= \mathbb{E} \left[\int_0^T \sup_{(u, \tilde{u}) \in C_s^n} \left(u^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{u}(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right] \\ &\geq \sup_{\{(H, \tilde{H}) | (H_s, \tilde{H}_s) \in C_s^n, s \in [0, T]\}} \mathbb{E} \left[\int_0^T \left(H_s^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right] \end{aligned} \quad (4.13)$$

$$= \sup_{\xi \in \mathcal{M}^n} \mathbb{E} \left[\int_0^T \left((H_s^\xi)^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s^\xi(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right], \quad (4.14)$$

with $\mathcal{M}^n := \{\xi \in \mathcal{Q} | (H_s^\xi, \tilde{H}_s^\xi) \in C_s^k, s \in [0, T]\}$, where the supremum in (4.13) is taken over pairs $(H, \tilde{H}) \in L_d^2(\mathcal{P}, d\mathbb{P} \times dt) \times L^2(\mathcal{P} \times \mathcal{B}(\mathbb{R}^k \setminus \{0\}), d\mathbb{P} \times dt \times \nu(dx))$.

Let us show next that the inequality in (4.13) is in fact an equality. It is well known (see for instance Theorem 2.4.9 in Zalinescu (2002)) that the subgradients of continuous and convex functions are non-empty so that the suprema in the dual representations of the functions g^n , $n \in \mathbb{N}$, are attained. Hence, we can apply a measurable selection theorem to the set

$$G^n := \left\{ (s, \omega, u, \tilde{u}) \left| g^n(s, \omega, H_s^X, \tilde{H}_s^X) - u^\top H_s^X - \int_{\mathbb{R}^k \setminus \{0\}} \tilde{u}(x) \tilde{H}_s^X(x) \nu(dx) + J_{C_s^n(\omega)}(u, \tilde{u}) = 0 \right. \right\},$$

obtaining $\mathcal{P} \times \mathcal{P} \otimes \mathcal{U}$ -measurable processes (U^n, \tilde{U}^n) such that, for every s , $(U_s^n, \tilde{U}_s^n) \in C_s^n$ and $g^n(s, H_s^X, \tilde{H}_s^X) = (U_s^n)^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{U}_s^n(x) \tilde{H}_s^X(x) \nu(dx)$. This implies (4.13) holds with equality, and yields the desired representation for $D^{(n)}$.

To see that we also get a representation for D let us first prove that the set C defined above (our natural candidate to satisfy (4.2)–(4.3)) is closed. Note that from (4.14) it follows that for any $X \in L^2(\mathcal{F}_T)$

$$\sup_{\xi \in \mathcal{S} \cap \mathcal{A}^n} \mathbb{E}[\xi X] = D_0^{(n)}(X) = \sup_{\xi \in \mathcal{M}^n} \mathbb{E} \left[\int_0^T \left((H_s^\xi)^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s^\xi(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right]. \quad (4.15)$$

As $\mathcal{S} \cap \mathcal{A}^n$ and \mathcal{M}^n are both convex and closed sets, we conclude from (4.15) $\mathcal{S} \cap \mathcal{A}^n = \mathcal{M}^n$. In particular, for $m \geq n$ we have $\mathcal{M}^n = \mathcal{M}^m \cap \mathcal{A}^n$. As there is a one-to-one correspondence between $\xi \in \mathcal{Q}$ and square-integrable predictable processes (H, \tilde{H}) this entails that

$$C^n = C^m \cap \left\{ (H, \tilde{H}) \in L^2(d\mathbb{P} \times dt) \times L^2(d\mathbb{P} \times dt \times \nu(dx)) \left| \sup_{t \in [0, T]} \{ |H_t|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{H}_t(x)|^2 \nu(dx) \} \leq n^2 \right. \right\}.$$

Hence, $d\mathbb{P} \times dt$ a.e.

$$C_t^n(\omega) = C_t^m(\omega) \cap \left\{ (h, \tilde{h}) \in \mathbb{R}^d \times L^2(\nu(dx)) \left| |h|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{h}(x)|^2 \nu(dx) \leq n^2 \right. \right\}.$$

Taking the union over all $m \in \mathbb{N}$ on the right-hand side of previous display yields

$$C_t^n(\omega) = C_t(\omega) \cap \left\{ (h, \tilde{h}) \in \mathbb{R}^d \times L^2(\nu(dx)) \left| |h|^2 + \int_{\mathbb{R}^k \setminus \{0\}} |\tilde{h}(x)|^2 \nu(dx) \leq n^2 \right. \right\}.$$

Since the sets $C_t^n(\omega)$, $n \in \mathbb{N}$, are closed in $\mathbb{R}^d \times L^2(\nu(dx))$, we have that also $C_t(\omega)$ is closed.

As g^n , $n \in \mathbb{N}$, are convex positively homogeneous driver functions it follows by Lemma 4.11 that $0 \in \text{int}(C^n)$. As $C^n \subset C$ we have thus that $0 \in \text{int}(C)$.

Finally, to show that C satisfies the desired representation (4.2)–(4.3) we note that $D_0(X)$ is equal to

$$\begin{aligned} \sup_{n \in \mathbb{N}} D_0^{(n)}(X) &= \sup_{n \in \mathbb{N}} \sup_{\{(H, \tilde{H}) | (H_s, \tilde{H}_s) \in C_s^n, s \in [0, T]\}} \mathbb{E} \left[\int_0^T \left(H_s^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right] \\ &= \sup_{\{(H, \tilde{H}) | (H_s, \tilde{H}_s) \in C_s, s \in [0, T]\}} \mathbb{E} \left[\int_0^T \left(H_s^\top H_s^X + \int_{\mathbb{R}^k \setminus \{0\}} \tilde{H}_s(x) \tilde{H}_s^X(x) \nu(dx) \right) ds \right] \\ &= \sup_{\{\xi \in \mathcal{Q} | (H_s^\xi, \tilde{H}_s^\xi) \in C_s, s \in [0, T]\}} \mathbb{E}[\xi X], \end{aligned}$$

where in the first and second line the suprema are taken over pairs $(H, \tilde{H}) \in L_d^2(\mathcal{P}, d\mathbb{P} \times dt) \times L^2(\mathcal{P} \times \mathcal{B}(\mathbb{R}^k \setminus \{0\}), d\mathbb{P} \times dt \times \nu(dx))$. This yields (4.2)–(4.3) for $s = 0$, and hence for all $s \in [0, T]$ by Remark 1.2(ii). Thus, the implication ‘ \Rightarrow ’ is shown, and the proof is complete. \square

5 Dynamic mean-deviation portfolio optimisation

We turn next to the stochastic optimisation problem of identifying a dynamic portfolio allocation strategy that maximizes the sum of the expected return and a penalty for its riskiness given in terms of a dynamic deviation measure of the final wealth achieved under this allocation strategy. Throughout this section we impose the following conditions:

Assumption 5.1 (i) The Lévy measure ν is such that $\nu(\{x \in \mathbb{R}^k \setminus \{0\} : \min_{i=1, \dots, k} x_i \leq -1\}) = 0$, and

$$\nu_2 := \int_{\mathbb{R}^k \setminus \{0\}} |x|^2 \nu(dx) < \infty. \quad (5.1)$$

(ii) D is a g -deviation measure with non-random, time-independent driver $\hat{g} : \mathbb{R}^d \times L^2(\nu(dx)) \rightarrow \mathbb{R}_+$.

Under (5.1), $L = (L_t^1, \dots, L_t^k)_{t \in [0, T]}$ with $L_t^j = \int_{[0, t] \times \mathbb{R}^k \setminus \{0\}} x_j \tilde{N}(ds \times dx)$, $j = 1, \dots, k$, where x_j is the j th coordinate of $x \in \mathbb{R}^k$, is a vector of pure-jump (\mathcal{F}_t) -martingales.

The financial market that we consider consists of a bank-account that pays interest at a fixed rate $r \geq 0$ and n risky stocks (with $1 \leq n \leq \min\{d, k\}$) with price processes $S^i = (S_t^i)_{t \in [0, T]}$, $i = 1, \dots, n$, satisfying the SDEs given by

$$\frac{dS_t^i}{S_{t-}^i} = \mu_i dt + \sum_{j=1}^d \sigma_{ij} dW_t^j + \sum_{j=1}^k \rho_{ij} dL_t^j, \quad t \in (0, T], \quad (5.2)$$

where $S_0^i = s_i \in \mathbb{R}_+ \setminus \{0\}$, $\mu_i \in \mathbb{R}$, $\sigma_{ij} \in \mathbb{R}_+$ and $\rho_{ij} \in \mathbb{R}_+$ such that $\sum_{j=1}^k \rho_{ij} \leq 1$ denote the rates of appreciation, the volatilities and the jump-sensitivities. By $\pi = (\pi^1, \dots, \pi^n)^\top$ we denote the dynamic allocation process that indicates the fraction of the total wealth that is invested in the stocks $1, \dots, n$ (that is, if $X^\pi(t-)$ denotes the wealth just before time t , $\pi_i(t)X^\pi(t-)$ is the cash amount invested in stock i at time t under allocation strategy π). We adopt the standard frictionless setting (no transaction costs, infinitely divisible stocks, continuous trading, *etc.*) and restrict to the case that short-sales and borrowing are not permitted, by only considering allocation processes $\pi = (\pi_t)_{t \in [0, T]}$ that take values in the set

$$\mathcal{B} = \left\{ x \in \mathbb{R}^{1 \times n} : \min_{i=1, \dots, n} x_i \geq 0, \sum_{i=1}^n x_i \leq 1 \right\}.$$

Such an allocation process π is said to be *admissible* if (i) π is predictable, (ii) the associated wealth process X^π is non-negative (that is, X^π satisfies the insolvency constraint $\inf_{t \in [0, T]} X_t^\pi \geq 0$) and (iii) π is a self-financing portfolio such that X^π satisfies the SDE (with $\mu = (\mu_1, \dots, \mu_n)^\top$, $\Sigma = (\sigma_{ij}) \in \mathbb{R}^{n \times d}$ and $R = (\rho_{ij}) \in \mathbb{R}^{n \times k}$) given by

$$\frac{dX_t^\pi}{X_{t-}^\pi} = [r + (\mu - r\mathbf{1})^\top \pi_t] dt + \pi_t^\top \Sigma dW_t + \pi_t^\top R dL_t, \quad t \in (0, T], \quad (5.3)$$

with initial wealth $X_0^\pi = x \in \mathbb{R}_+ \setminus \{0\}$, where $\mathbf{1} \in \mathbb{R}^{n \times 1}$ denotes the column vector of ones. We denote by Π the collection of admissible allocation strategies and let $\gamma > 0$ denote a risk-aversion parameter. To a given allocation strategy $\pi \in \Pi$ we associate the following dynamic performance criterion:

$$J_t^\pi := E[X_T^\pi | \mathcal{F}_t] - \gamma D_t(X_T^\pi), \quad t \in [0, T]. \quad (5.4)$$

Due to the fact that, unlike the conditional expectation, $D_t(X)$ is a non-linear function of X , the Dynamic Programming Principle is not satisfied for this objective. There is a growing literature

exploring alternative solution approaches to dynamic optimisation problems for which the Dynamic Programming Principle is not applicable. One alternative dynamic solution concept is that of subgame-perfect Nash equilibrium—in such a game-theoretic approach the problem (5.4) may informally be seen as a (non-cooperative) game with infinitely many players, one for each time t , which may be interpreted in terms of the changing preferences of one person over time; see Ekeland and Pirvu (2008) and Björk and Murgoci (2010) for background, and see Basak and Chabakauri (2010), Björk and Murgoci (2010), Wang and Forsyth (2011), Czichowsky (2013), Björk *et al.* (2014), Bensoussan *et al.* (2014), and references therein, for studies of dynamic mean-variance portfolio optimisation problems. Following Ekeland and Pirvu (2008) and Björk and Murgoci (2010) we have the following formalisation of this equilibrium solution concept in our setting:

Definition 5.2 (i) *An allocation strategy $\pi^* \in \Pi$ is an equilibrium policy for the dynamic mean-deviation problem with objective (5.4) if*

$$\liminf_{h \searrow 0} \frac{J_t^{\pi^*} - J_t^{\pi(h)}}{h} \geq 0 \quad (5.5)$$

for any $t \in [0, T)$ and any policy $\pi(h) \in \Pi$ satisfying, for some $\pi \in \Pi$,

$$\pi(h)_s = \pi_s I_{[t, t+h)}(s) + \pi_s^* I_{[t+h, T]}(s), \quad s \in [t, T].$$

(ii) *An equilibrium policy π^* is of feedback type if, for some feedback function $\pi_* : [0, T] \times \mathbb{R}_+ \rightarrow \mathcal{B}$ such that (5.3) with π_t replaced by $\pi_*(t, X_{t-})$ has a unique solution $X^* = (X_t^*)_{t \in [0, T]}$, we have*

$$\pi_t^* = \pi_*(t, X_{t-}^*), \quad t \in [0, T],$$

with $X_{0-}^* = X_0^*$.

For a given equilibrium policy $\pi^* = (\pi_t^*)_{t \in [0, T]}$ of feedback type we have by the Markov property that $J_t^{\pi^*} = V(t, X_t^*)$ and $\mathbb{E}[X_T^* | \mathcal{F}_t] = h(t, X_t^*)$, $t \in [0, T]$, for some functions $V : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $h : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Furthermore, if h is sufficiently regular (e.g., $h \in C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R}_+)$ and $h' \equiv \frac{\partial h}{\partial x}$ is bounded) we find by an application of Itô's lemma that the representing pair of X_T^* is given by

$$\begin{aligned} H_s^{X_T^*} &= a^{*,h}(s, X_{s-}^*), \quad \tilde{H}_s^{X_T^*}(y) = b^{*,h}(s, X_{s-}^*, y), \quad s \in [0, T], \quad y \in \mathbb{R}^k \setminus \{0\}, \text{ with} \\ a^{*,h}(s, x) &:= h'(s, x) x \pi_*(s, x)^\top \Sigma, \quad b^{*,h}(s, x, y) := h(s, x + x \pi_*(s, x)^\top R y) - h(s, x), \end{aligned}$$

so that $D_t(X_T^*)$, $t \in [0, T]$, takes the form $D_t(X_T^*) = \tilde{D}_{t, X_t^*}(X_T^*)$, where

$$\tilde{D}_{t,x}(X_T^*) = \mathbb{E}_{t,x} \left[\int_t^T \hat{g}(a^{*,h}(s, X_{s-}^*), b^{*,h}(s, X_{s-}^*, \cdot)) ds \right], \quad (t, x) \in [0, T] \times \mathbb{R}_+, \quad (5.6)$$

with $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot | X_t^* = x]$. To any vector $\pi \in \mathcal{B}$ we associate the operators $\mathcal{L}^\pi : f \mapsto \mathcal{L}^\pi f$ and $\mathcal{G}^\pi : f \mapsto \mathcal{G}^\pi f$ that map $C^{0,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$ to $C^{0,0}(\mathbb{R}_+, \mathbb{R})$ and are given by

$$\mathcal{L}^\pi f(t, x) = \mu_\pi x f'(t, x) + \frac{\sigma_\pi^2}{2} x^2 f''(t, x) + \int_{\mathbb{R}^k \setminus \{0\}} [f(t, x + x \pi^\top R y) - f(t, x) - x \pi^\top R y f'(t, x)] \nu(dy), \quad (5.7)$$

$$\mathcal{G}^\pi f(t, x) = \hat{g}(x f'(t, x) \pi^\top \Sigma, \delta_{x \pi^\top R I} f(t, x)), \quad (5.8)$$

for $(t, x) \in [0, T] \times \mathbb{R}_+$, where $\delta_y f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $I : \mathbb{R}^{k \times 1} \rightarrow \mathbb{R}^{k \times 1}$ are given by

$$\delta_y f(x) = f(t, y + x) - f(t, x), \quad I(z) = z, \quad z \in \mathbb{R}^{k \times 1}, x \in \mathbb{R}_+, y \in \mathbb{R},$$

and where

$$\mu_\pi = r + (\mu - r\mathbf{1})^\top \pi, \quad \sigma_\pi^2 = \pi^\top \Sigma \Sigma^\top \pi, \quad \pi \in \mathcal{B}.$$

Given the form of the objective and Definition 5.2 we are led to consider the *extended Hamilton-Jacobi-Bellman equation* for a triplet (π_*, V, h) of a feedback function π_* , the corresponding value function V and auxiliary function h given by (denoting $\dot{V} = \frac{\partial V}{\partial t}$):

$$\dot{V}(t, x) + \sup_{\pi \in \mathcal{B}} \{ \mathcal{L}^\pi V(t, x) - \gamma \mathcal{G}^\pi h(t, x) \} = 0, \quad (t, x) \in [0, T) \times \mathbb{R}_+ \setminus \{0\}, \quad (5.9)$$

$$\dot{h}(t, x) + \mathcal{L}^{\pi_*(t, x)} h(t, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}_+ \setminus \{0\}, \quad (5.10)$$

$$V(T, x) = h(T, x) = x, \quad x \in \mathbb{R}_+, \quad (5.11)$$

$$V(t, 0) = h(t, 0) = 0, \quad t \in [0, T], \quad (5.12)$$

where, for any $t \in [0, T]$ and $x \in \mathbb{R}_+$, $\pi_*(t, x)$ is a maximiser of the supremum in (5.9) (note that the continuity of $\mathcal{L}^\pi V(t, x)$ and $\mathcal{G}^\pi h(t, x)$ in π for each fixed $t \in [0, T]$ and $x \in \mathbb{R}_+ \setminus \{0\}$ in conjunction with the compactness of \mathcal{B} guarantees that the maximum in (5.9) is attained).

We have the following verification result:

Theorem 5.3 *Let (π_*, h, V) be a triplet satisfying the extended HJB equation (5.9)–(5.12), let X^* be the unique solution of (5.3) with π_t replaced by $\pi_*(t, X_{t-}^*)$ and define $\pi_t^* = \pi_*(t, X_{t-}^*)$, $t \in [0, T]$. Assume $h, V \in C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$ with h', V' bounded and that $\pi^* = (\pi_t^*)_{t \in [0, T]} \in \Pi$. Then π^* is an equilibrium policy of feedback type and h and V are given by $V(t, x) = \mathbb{E}_{t,x}[X_T^{\pi^*}] - \gamma \tilde{D}_{t,x}(X_T^{\pi^*})$ and $h(t, x) = \mathbb{E}_{t,x}[X_T^{\pi^*}]$ for $(t, x) \in [0, T] \times \mathbb{R}_+$.*

Proof. We first verify the stochastic representations. Let $\pi = (\pi_s)_{s \in [0, T]} \in \Pi$, $t \in [0, T)$ and $\tau \in (0, T - t)$ be given and denote $\Xi^{\pi, V, h}(s, x) := (\dot{V} + \mathcal{L}^{\pi_s} V - \gamma \mathcal{G}^{\pi_s} h)(s, x)$, $a^{\pi, V}(s, x) := V'(s, x) x \pi_s^\top \Sigma$ and $b^{\pi, V}(s, x, y) = V(s, x + x \pi_s^\top R y) - V(s, x)$. An application of Itô's lemma to $V(t + \tau, X_{t+\tau}^\pi)$ shows that

$$\begin{aligned} V(t + \tau, X_{t+\tau}^\pi) - V(t, X_t^\pi) - \gamma \int_t^{t+\tau} \mathcal{G}^{\pi_s} h(s, X_{s-}^\pi) ds &= \int_t^{t+\tau} \Xi^{\pi, V, h}(s, X_{s-}^\pi) ds \\ &+ \int_t^{t+\tau} a^{\pi, V}(s, X_{s-}^\pi) dW_s + \int_{(t, t+\tau] \times \mathbb{R}^k \setminus \{0\}} b^{\pi, V}(s, X_{s-}^\pi, y) \tilde{N}(ds \times dy). \end{aligned} \quad (5.13)$$

Similarly, it follows $h(t + \tau, X_{t+\tau}^\pi)$ satisfies (5.13) with V , $\mathcal{G}^{\pi_s} h$ and $\Xi^{\pi, V, h}$ replaced by h , 0 and $\Xi^{\pi, h, 0}$, respectively. In particular, choosing π equal to π^* , and taking expectations, the three terms on the right-hand side of (5.13) vanish in view of (5.9), the fact that $\pi_*(t, x)$ is a maximiser in (5.9) and as the stochastic integrals are martingales (in view of the boundedness of V' , h' and \mathcal{B}). Then, letting $\tau \nearrow T - t$ and using the boundary conditions (5.11), we obtain the asserted stochastic representations of h and V .

Next we turn to the proof that π^* is an equilibrium solution. By an application of the tower-property of conditional expectation and (D6) we have for any $\pi \in \Pi$, $t \in [0, T)$ and $\tau \in (0, T - t)$

$$\begin{aligned} J_t^\pi &= \mathbb{E}[\mathbb{E}[X_T^\pi | \mathcal{F}_{t+\tau}] | \mathcal{F}_t] - \gamma \mathbb{E}[D_{t+\tau}(X_T^\pi) | \mathcal{F}_t] - \gamma D_t(\mathbb{E}[X_T^\pi | \mathcal{F}_{t+\tau}]) \\ &= \mathbb{E}[J_{t+\tau}^\pi | \mathcal{F}_t] - \gamma D_t(\mathbb{E}[X_T^\pi | \mathcal{F}_{t+\tau}]). \end{aligned} \quad (5.14)$$

Fixing $(\epsilon_n)_n$, $\epsilon_n \searrow 0$, and strategies $\pi_n := \pi(\epsilon_n) \in \Pi$ as in Definition 5.2 (with π^* as asserted in the theorem) and noting that the Markov property (which is in force as π^* is a feedback strategy) implies

$$J_{t+\epsilon_n}^{\pi_n} = V(t + \epsilon_n, X_{t+\epsilon_n}^{\pi_n}), \quad \mathbb{E}[X_T^{\pi_n} | \mathcal{F}_{t+\epsilon_n}] = h(t + \epsilon_n, X_{t+\epsilon_n}^{\pi_n}), \quad (5.15)$$

and that (5.15) remains valid with π_n replaced by π^* , we have from (5.13) and (5.14) and the fact $D_t(h(t + \epsilon_n, X_{t+\epsilon_n}^{\pi_n})) = \mathbb{E} \left[\int_t^{t+\epsilon_n} \hat{g}(a_s^{\pi_n, h}, b_s^{\pi_n, h}) ds \middle| \mathcal{F}_t \right]$ that

$$J_t^{\pi^*} - J_t^{\pi_n} = \mathbb{E} \left[\int_t^{t+\epsilon_n} [\Xi^{\pi^*, V, h}(s, X_{s-}^{\pi^*}) - \Xi^{\pi_n, V, h}(s, X_{s-}^{\pi_n})] ds \middle| \mathcal{F}_t \right]. \quad (5.16)$$

Since $\Xi^{\pi^*(s, x), V, h}(s, x) = 0$ and $\Xi^{\pi_n, V, h}(s, x) \leq 0$ for $s \in [0, T]$, $x \in \mathbb{R}_+$, (by (5.9) and the fact that $\pi_*(t, x)$ is the maximiser in (5.9)) we have $\liminf_{n \rightarrow \infty} (J_t^{\pi^*} - J_t^{\pi_n})/\epsilon_n \geq 0$, and the proof is complete. \square

We next identify an explicit equilibrium policy for the mean-deviation portfolio optimisation problem, under the following regularity assumption on Σ , R and \hat{g} , assumed to be in force in the sequel:

Assumption 5.4 For some countable set A and any $a \in [0, \gamma^{-1}] \setminus A$, the function $T_a : \mathcal{B} \rightarrow \mathbb{R}$ given by

$$T_a(c) := a(\mu - r\mathbf{1})^\top c - \hat{g}(c^\top \Sigma, c^\top R I), \quad c \in \mathcal{B}, \quad (5.17)$$

achieves its maximum over $\partial \mathcal{B}$ at a unique $c^* \in \partial \mathcal{B}^{**}$

To define the optimal policy we deploy the following auxiliary result:

Lemma 5.5 For any $f : [0, T] \rightarrow \mathcal{B}$ denote by $A_f, d_f, b_f, F_f : [0, T] \rightarrow \mathbb{R}$ the functions given by

$$b_f(t) := \exp \left(\int_t^T \{r + (\mu - r\mathbf{1})^\top f(s)\} ds \right), \quad (5.18)$$

$$d_f(t) := b_f(t) \int_t^T \hat{g}(f(s)^\top \Sigma, f(s)^\top R I) ds, \quad (5.19)$$

$$A_f(t) := \gamma^{-1} - (b_f(t))^{-1} d_f(t), \quad (5.20)$$

$$F_f(t) := A_{C_f}(t), \quad \text{with} \quad (5.21)$$

$$C_f(t) := \begin{cases} \arg \sup_{c \in \partial \mathcal{B}} \{T_{f(t)}(c)\}, & \text{if } f(t) \notin A, \\ \text{Centroid}(\arg \sup_{c \in \partial \mathcal{B}} \{T_{f(t)}(c)\}), & \text{if } f(t) \in A, \end{cases} \quad (5.22)$$

where for any Borel set $A' \subset \mathbb{R}^d$, $\text{Centroid}(A')$ is equal to the mean of $U \sim \text{Unif}(A')$. Then there exists a continuous non-decreasing function $a^* : [0, T] \rightarrow \mathbb{R}_+$ such that $a^* = F_{a^*}$.

The proof of Lemma 5.5 is provided below. With this result in hand we identify an equilibrium policy as follows:

Theorem 5.6 With $T_a(c)$ and a^* given in (5.17) and in Lemma 5.5, we let $s(a) := \sup_{c \in \partial \mathcal{B}} T_a(c)$, $a_- := \sup\{a \in [0, \gamma^{-1}] : s(a) \leq 0\}$, and $t^* := \sup\{t \in [0, T] : a^*(t) \leq a_-\}$ (where $\sup \emptyset := -\infty$).

(i) If $s(1/\gamma) \leq 0$ then $\pi^* \equiv 0$ with value-function given by $V(t, x) = x \exp(r(T - t))$ for $(t, x) \in [0, T] \times \mathbb{R}_+$.

(ii) If $s(1/\gamma) > 0$ define the function $C^* : [0, T] \rightarrow \mathcal{B}$ by

$$C^*(t) = \begin{cases} C_{a^*(t)}, & \text{if } t \in [t^* \vee 0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

where $C_{a^*(t)}$ is given in (5.22) with $f = a^*$. Then $\pi^* = C^*$ is an equilibrium policy with value function given by $V(t, x) = x(b_{C^*}(t) - \gamma d_{C^*}(t))$ for $(t, x) \in [0, T] \times \mathbb{R}_+$, where b_{C^*} and d_{C^*} are given in (5.18) and (5.19) with $f = C^*$.

^{**} $\partial \mathcal{B}$ denotes the boundary of \mathcal{B} , that is, $\partial \mathcal{B} = \text{cl}(\mathcal{B}) \setminus \text{int}(\mathcal{B})$ where $\text{cl}(\mathcal{B})$ and $\text{int}(\mathcal{B})$ denote the closure and interior of \mathcal{B} .

Remark 5.7 Under the equilibrium policy π^* given in Theorem 5.6 it is optimal to invest in the n stocks according to the proportions $C^* = (C_1^*, \dots, C_n^*)$ of the current wealth, which are non-random functions of t only. Hence, it is optimal to invest at time t an amount $X^{\pi^*}(t-)C_i^*(t)$ in stock i , $i = 1, \dots, n$.

Proof of Theorem 5.6. The proof consists in verifying that the triplet (π_*, V, h) , with π_* and V as stated and with $h : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $h(t, x) = x b_{C^*}(t)$, satisfies the extended HJB equation (5.9)–(5.12); the assertions then follow by an application of Theorem 5.3.

(i) Once we verify that the supremum in (5.9) is attained at $\pi_* \equiv 0$ it is easily checked that V and h are equal and satisfy (5.9)–(5.12), using that g is positively homogeneous. To see that the former is the case note that the left-hand side of (5.9) is equal to $x \exp(r(T-t)) [-r + \gamma \sup_{c \in \mathcal{B}} T_{1/\gamma}(c)]$; since $s(1/\gamma) \leq 0$, the latter supremum is zero and it is attained at $c = 0$ (as $T_{1/\gamma}(0) = 0$).

(ii) Assume for the moment that the supremum in (5.9) is attained at π^* . Then the positive homogeneity of g and the fact (which is straightforward to verify) that functions b_{C^*} and d_{C^*} satisfy the system of equations

$$\begin{aligned} \dot{b} + (r + \mu_{C^*})b &= 0, \quad t \in [0, T], \quad b(T) = 1, \\ \dot{d} + (r + \mu_{C^*})d + b\hat{g}((C^*)^\top \Sigma, (C^*)^\top R I) &= 0, \quad t \in [0, T], \quad d(T) = 0, \end{aligned}$$

where as before $I : \mathbb{R}^{k \times 1} \rightarrow \mathbb{R}^{k \times 1}$ is given by $I(y) = y$, imply that h and V satisfy (5.9)–(5.12).

Next we verify that the supremum in (5.9) is attained at π_* . Inserting the forms of h and V and using that $\gamma \inf_{t \in [0, T]} b_{C^*}(t) > 0$ we have for any $t \in [0, T]$ that

$$\begin{aligned} \arg \sup_{\pi \in \mathcal{B}} \{ \mathcal{L}^\pi V(t, x) - \gamma \mathcal{G}^\pi h(t, x) \} &= \arg \sup_{\pi \in \mathcal{B}} \{ \mu_\pi (b_{C^*}(t) - \gamma d_{C^*}(t)) - \gamma b_{C^*}(t) \hat{g}(\pi^\top \Sigma, \pi^\top R I) \} \\ &= \arg \sup_{\pi \in \mathcal{B}} \{ \mu_\pi A_{C^*}(t) - \hat{g}(\pi^\top \Sigma, \pi^\top R I) \}. \end{aligned} \quad (5.23)$$

If $t \leq t^*$, then $A_{C^*}(t) = a_-$ so that $s(A_{C^*}(t)) \leq 0$ and 0 is included in the argsup in (5.23), while if $t > t^*$, then $A_{C^*}(t) > a_-$ and we have that $s(A_{C^*}(t)) = \sup_{\pi \in \mathcal{B}} \{ (\mu_\pi - r) A_{C^*}(t) - \hat{g}(\pi^\top \Sigma, \pi^\top R I) \} > 0$ is attained at $\pi = C_{A_{C^*}}(t) = C_{a^*}(t) = C^*(t)$. \square

Proof of Lemma 5.5. The proof relies on an application of Schauder's fixed point theorem^{††} to the map $F : \mathbb{A} \rightarrow C([0, T], \mathbb{R})$ given by $f \mapsto F_f$, where \mathbb{A} denotes the set of continuous functions $f \in C([0, T], \mathbb{R})$ that are such that (a) $f(T) = \gamma^{-1}$ and (b) for all $s, t \in [0, T]$ with $s \leq t$ we have $f(t) - f(s) \in [\chi_-(t-s), \chi_+(t-s)]$ where

$$\chi_+ := \sup \{ \hat{g}(c^\top \Sigma, c^\top R I) : c \in \partial \mathcal{B} \}, \quad \chi_- := \inf \{ \hat{g}(c^\top \Sigma, c^\top R I) : c \in \partial \mathcal{B} \}.$$

We note that both χ_+ and χ_- are strictly positive, by positivity of the driver function \hat{g} . It is straightforward to verify that F maps \mathbb{A} to \mathbb{A} and that the set \mathbb{A} is a non-empty, closed, bounded and convex subset of $C([0, T], \mathbb{R})$. Since F is compact (as we prove below), Schauder's fixed point theorem yields that there exists an element $a^* \in \mathbb{A}$ such that $a^* = F_{a^*}$.

We next prove that F is compact by showing that (i) F is continuous (with respect to the supremum norm on $[0, T]$) and (ii) the set $F(\mathbb{A}) = \{F_f : f \in \mathbb{A}\}$ is relatively compact in $C([0, T], \mathbb{R})$.

(i) Let $(f_n)_n \subset \mathbb{A}$ converge to $f \in \mathbb{A}$ in the supremum-norm. Then we have that $T_{f_n(t)}(c) \rightarrow T_{f(t)}(c)$ as $n \rightarrow \infty$ uniformly in $t \in [0, T]$ for any $c \in \partial \mathcal{B}$, and $\sup_{c \in \partial \mathcal{B}} T_{f_n(t)}(c) \rightarrow \sup_{c \in \partial \mathcal{B}} T_{f(t)}(c)$ for any $t \in [0, T]$. As $(f_n)_n$ and f are strictly monotone increasing and Assumption 5.4 is in force, we have for all but countably many t that $T_{f_n(t)}(c)$ and $T_{f(t)}(c)$ attain their maxima over $\partial \mathcal{B}$ at unique c . Thus, it follows that $\arg \sup_{c \in \partial \mathcal{B}} T_{f_n(t)}(c) \rightarrow \arg \sup_{c \in \partial \mathcal{B}} T_{f(t)}(c)$, for a.e. $t \in [0, T]$. Hence, by the dominated

^{††}see e.g. Theorem 1.C in Zeidler (1995)

convergence theorem $F_{f_n}(t) = A_{C_{f_n}}(t) \rightarrow A_{C_f}(t) = F_f(t)$ for any $t \in [0, T]$. Since the functions $A_{C_{f_n}}$ and A_{C_f} are non-decreasing, the convergence $F_{f_n} \rightarrow F_f$ holds in the supremum norm.

(ii) Using the boundedness of \mathcal{B} and the continuity of \hat{g} it is straightforward to verify that the collection of functions $F(\mathbb{A})$ is equi-continuous. Hence we have by an application of the Arzela-Ascoli theorem^{††} that for any sequence $(A^{(n)})_n \subset F(\mathbb{A})$ there exists a continuous function $A^* : [0, T] \rightarrow \mathbb{R}$ such that, along a subsequence (n_k) , $(A^{(n_k)})_k$ converges uniformly to A^* , hence establishing that $F(\mathbb{A})$ is relatively compact. \square

Example 5.8 (i) For driver function $\hat{g} = g_1$ (given in Example 2.10 with $\lambda = 1$) and for $a \in \mathbb{R}_+$ we have that $T_a(c)$ in (5.17) is given by

$$T_a(c) = a(\mu - r\mathbf{1})^\top c - \sqrt{c^\top \Sigma \Sigma^\top c + c^\top R R^\top c \nu_2}.$$

If $\Sigma \Sigma^\top + R R^\top \nu_2$ is invertible, then it is straightforward to verify that Assumption 5.4 is satisfied.

(ii) Let us identify explicitly the equilibrium portfolio allocation strategy given in Theorem 5.6 in the case the driver function \hat{g} is as in part (i) and we have 2 risky assets ($n = 2$), whose dynamics we suppose are given by (5.2) with $d = k = 2$, $\mu_1 > \mu_2 > r$, $r \geq 0$ and $s_{12} := (\Sigma^2 + R^2 \nu_2)_{12} < 0$. In terms of $s_i^2 := (\Sigma^2 + R^2 \nu_2)_{ii}$, $i = 1, 2$, let us denote

$$\begin{aligned} d_+ &:= s_1^2 + s_2^2 - 2s_{12}, \quad e_+ := s_{12} - s_2^2, \\ c_+(a) &:= -\frac{e_+}{d_+} + \sqrt{\left(\frac{e_+}{d_+}\right)^2 - \eta(a)}, \quad \eta(a) := \frac{a^2(\mu_1 - \mu_2)^2 s_2^2 - e_+^2}{d_+(a^2(\mu_1 - \mu_2)^2 - d_+)}, \end{aligned}$$

for $a \in [0, \sqrt{d_+}/(\mu_1 - \mu_2)]$. By convexity of g it follows that the supremum of $\tilde{T}(c) := T_a((c, 1 - c)) = a(\mu_2 - r) + a(\mu_1 - \mu_2)c - \sqrt{d_+ c^2 + 2e_+ c + s_2^2}$ over $c \in \mathbb{R}$ is attained at the c satisfying $\tilde{T}'(c) = 0 \Leftrightarrow c = c_+(a)$ and we have

$$\tilde{T}'(1) > 0 \Leftrightarrow a > a_+ := \frac{1}{\mu_1 - \mu_2} \left(\frac{s_1^2 - s_{12}}{\sqrt{s_1^2}} \right).$$

As a consequence, the equilibrium allocation strategy $\pi^* = (\pi_t^*)_{t \in [0, T]}$ in Theorem 5.6 is given as follows:

$$\pi_t^* = C^*(t) = \begin{cases} (1, 0), & \text{if } a_*(t) > a_- \vee a_+, \\ (c_+(a_*(t)), 1 - c_+(a_*(t))), & \text{if } a_- < a_*(t) \leq a_- \vee a_+, \\ (0, 0), & \text{if } a_*(t) \leq a_-, \end{cases}$$

where a_- and $a_*(t)$ are as in Theorem 5.6. Hence, if the risk-aversion parameter γ is sufficiently small and/or t is sufficiently close to the horizon T the equilibrium strategy is to be fully invested in risky asset 1, which has the highest expected return; at times t further away from the horizon or for higher risk-aversion parameter, the dynamic deviation penalty term starts to play a more important role and the policy is to invest part of the wealth into asset 2, while, if γ is sufficiently large or t is sufficiently small, the equilibrium strategy is to invest all the wealth in the bank account.

(iii) Restricting next to the case of a single risky asset ($n = 1$) with $d = k = 1$, $\mu := \mu_1 > r$, we find by a direct calculation that the value function V in Theorem 5.6 and the auxiliary function h are explicitly given in terms of

$$t^* = \left(T + \frac{1}{\mu - r} - \frac{1}{\gamma \sqrt{\Sigma^2 + R^2 \nu_2}} \right) \wedge T$$

^{††}see e.g. p.35 in Zeidler (1995)

by $V(t, x) = V(t^* \wedge T, x \exp\{r(t^* \wedge T - t)\})$ and $h(t, x) = h(t^* \wedge T, x \exp\{r(t^* \wedge T - t)\})$ for $t \in [0, t^* \wedge T)$ and

$$V(t, x) = h(t, x)[1 - (T - t)\gamma\sqrt{\Sigma^2 + R^2\nu_2}], \quad h(t, x) = x \exp\{\mu(T - t)\}, \quad t \in [t^* \wedge T, T],$$

where the equilibrium policy π^* is given by

$$\pi_t^* = C^*(t) = \begin{cases} 1, & \text{if } a(t) = \frac{1}{\gamma} \frac{1}{1 + (\mu - r)(T - t)} > \frac{\sqrt{\Sigma^2 + R^2\nu_2}}{\mu - r} = a_- \Leftrightarrow t \in (t^* \wedge T, T], \\ 0, & \text{if } a(t) \leq a_- \Leftrightarrow t \in [0, t^* \wedge T]. \end{cases}$$

To see that π^* takes this form we observe that $t \leq t^*$ holds precisely if $(\mu - r - \gamma\sqrt{\Sigma^2 + R^2\nu_2}) - (\mu - r)\gamma(T - t)\sqrt{\Sigma^2 + R^2\nu_2} \leq 0 \Leftrightarrow 0 \in \arg. \sup_{\pi \in [0, 1]} \{(\mathcal{L}^\pi V)(t, x) - \gamma(\mathcal{G}^\pi h)(t, x)\}$, where \mathcal{L}^π and \mathcal{G}^π are given in (5.7) and (5.8).

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